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Number theory

On the cohomology of semi-stable p-adic Galois representations

Sur la cohomologie des représentations galoisiennes p-adiques semi-stables

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ABSTRACT

Let K be a field of characteristic 0 complete with respect to a non-trivial discrete valuation with perfect residue field k of characteristic $p > 0$. Let V be a p-adic representation of the absolute Galois group of K. We compute explicitly Kato's filtration on the continuous cohomology group $H^1(K, V)$. When k is finite, we give a simple proof of Hyodo's celebrated result $H^1_g(K, V) = H^1_{st}(K, V)$ when V is a potentially semi-stable Galois representation.

RÉSUMÉ

Soit K un corps de caractéristique 0 complet pour une valuation discrète non triviale à corps résiduel parfait k de caractéristique $p > 0$. Soit V une représentation p-adique du groupe de Galois absolu de K. On calcule explicitement la filtration de Kato sur le groupe de cohomologie continue $H^1(K, V)$. Lorsque k est fini, on en déduit une preuve simple du résultat bien connu de Hyodo qui dit que, si V est potentiellement semi-stable, alors $H^1_g(K, V) = H^1_{st}(K, V)$.

1. Explicit computation of Kato's filtration on the Galois cohomology

We fix a prime number p and a perfect field k of characteristic $p > 0$. We denote K_0 the fraction field of Witt vectors with coefficients in k and we fix a finite totally ramified extension K of K_0. We choose an algebraic closure $\overline{K}$ of K and set $G_K = \mathrm{Gal}(\overline{K}/K)$.

The topological $\mathbb{Q}_p$-vector spaces V equipped with a linear and continuous action of G_K form, in an obvious way, a $\mathbb{Q}_p$-linear additive exact category $C_{\mathbb{Q}_p}(G_K)$. For any object V of this category and $i \in \mathbb{N}$, we denote $H^i(K, V) = H^i_{\mathrm{cont}}(G_K, V)$ the i-th group of continuous cohomology (see Tate [7, §2]). Given a short exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

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of $C_{\mathbb{Q}_p}(G_K)$, we have an obvious exact sequence[1]

$$0 \to H^0\big(K, V'\big) \to H^0(K, V) \to H^0\big(K, V''\big) \to H^1\big(K, V'\big) \to H^1(K, V) \to H^1\big(K, V''\big)$$

With the extension $\overline{K}/K$ are associated the p-adic completion C of $\overline{K}$ and the usual rings of p-adic periods B_{dR}, B_{cris} and B_{st} which are topological rings equipped with a $\mathbb{Q}_p$-linear and continuous action of $G_K = \mathrm{Gal}(\overline{K}/K)$ (cf. [3] or [4]).

Let's choose a non-zero topologically nilpotent element π of K and a sequence $\varpi = (\varpi^{(n)})_{n\in\mathbb{N}}$ of elements of $\overline{K}$ such that $\varpi^{(0)} = \pi$ and $(\varpi^{(n+1)})^p = \varpi^{(n)}$ for all $n \in \mathbb{N}$. Recall that this choice defines an element $u = \log[\varpi]$ of B_{st} and that we can view also u as an element of B_{dR} by deciding that $\log(\pi) = 0$ (then we identify u to $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{([u]-1)^n}{n\pi^n}$). With these choices,

$$B_{\mathrm{st}} = B_{\mathrm{cris}}[u]$$

is a polynomial algebra in u with coefficients in B_{cris} and is a G_K-stable subring of B_{dR}. Moreover B_{dR} is a field containing K and, if we denote K_0 the fraction field of the ring $W(k)$ of Witt vectors with coefficients in k, we have:

$$H^0\big(K, B_{\mathrm{dR}}^+\big) = H^0(K, B_{\mathrm{dR}}) = K \quad \text{and} \quad H^0(K, B_{\mathrm{cris}}) = H^0(K, B_{\mathrm{st}}) = K_0.$$

The ring B_{st} is equipped with an endomorphism φ semi-linear with respect to the absolute Frobenius on K_0 and the B_{cris}-derivation $N = -\mathrm{d}/\mathrm{d}u$. The operators φ and N commute with G_K and satisfy $N\varphi = p\varphi N$. Therefore B_{cris} is the subring of B_{st} kernel of N and we define the ring B_e as the subring of B_{cris}, which is fixed by $\varphi - 1$. We have short exact sequences:

$$0 \longrightarrow B_{\mathrm{cris}} \longrightarrow B_{\mathrm{st}} \xrightarrow{N} B_{\mathrm{st}} \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow B_e \longrightarrow B_{\mathrm{cris}} \xrightarrow{\varphi-1} B_{\mathrm{cris}} \longrightarrow 0. \tag{2}$$

We set $\widetilde{B}_{\mathrm{dR}} = B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ and, for all $b \in B_{\mathrm{dR}}$, we denote $\tilde{b}$ its image in $\widetilde{B}_{\mathrm{dR}}$. The *fundamental exact sequence of p-adic Hodge theory* is the exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow \widetilde{B}_{\mathrm{dR}} \longrightarrow 0 \tag{3}$$

where $B_e \mapsto \widetilde{B}_{\mathrm{dR}}$ is the compositum of the inclusion $B_e \subset B_{\mathrm{dR}}$ with the projection $B_{\mathrm{dR}} \to \widetilde{B}_{\mathrm{dR}}$.

We now consider a p-adic Galois representation, i.e. a finite-dimensional $\mathbb{Q}_p$-vector space V equipped with a continuous linear action of G_K. Recall that we have a natural filtration by sub-$\mathbb{Q}_p$-vector spaces on $H^1(K, V)$, the *Kato's filtration*:

$$0 \subset H_e^1(K, V) \subset H_f^1(K, V) \subset H_{\mathrm{st}}^1(K, V) \subset H_g^1(K, V) \subset H^1(K, V)$$

where

$$H_e^1(K, V) = \ker\big(H^1(K, V) \longrightarrow H^1(K, B_e \otimes_{\mathbb{Q}_p} V)\big),$$
$$H_f^1(K, V) = \ker\big(H^1(K, V) \longrightarrow H^1(K, B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)\big),$$
$$H_{\mathrm{st}}^1(K, V) = \ker\big(H^1(K, V) \longrightarrow H^1(K, B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)\big),$$
$$H_g^1(K, V) = \ker\big(H^1(K, V) \longrightarrow H^1(K, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)\big).$$

We want to compute these cohomology groups. Recall that [5, Chap. I, §2.2.1] the *tangent space of V* is the K-vector space:

$$t_V = H^0(K, \widetilde{B}_{\mathrm{dR}} \otimes V).$$

We let N and φ act on $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$ via $N(b \otimes v) = Nb \otimes v$ and $\varphi(b \otimes v) = \varphi b \otimes v$. These actions commute with the action of G_K, hence N and φ act also on

$$D = D_{\mathrm{st}}(V) = H^0(K, B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)$$

which is a finite-dimensional K_0-vector space.

[1] If there is a (set-theoretic) continuous splitting of the projection $V \to V''$, we even get the usual long exact sequence (loc. cit.), but we will not use this fact.

1.1. $H^1_e(K,V)$

Tensoring with V, we get from (3) a short exact sequence

$$0 \longrightarrow V \longrightarrow B_e \otimes V \longrightarrow \widetilde{B}_{\mathrm{dR}} \otimes V \longrightarrow 0$$

inducing a long exact sequence

$$0 \to H^0(K,V) \to D_{N=0,\varphi=1} \to t_V \longrightarrow H^1_e(K,V) \longrightarrow 0 \tag{S_e}$$

where

$$D_{N=0,\varphi=1} = H^0(K, B_e \otimes V) = \{x \in D \mid Nx = 0,\ \varphi(x) = x\}.$$

1.2. $H^1_f(K,V)$

Consider the map $B_{\mathrm{cris}} \to B_{\mathrm{cris}} \oplus \widetilde{B}_{\mathrm{dR}}$ sending b to $(\varphi b - b, \tilde{b})$. From the exactness of (2) and (3), we get the exactness of

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{\mathrm{cris}} \longrightarrow B_{\mathrm{cris}} \oplus \widetilde{B}_{\mathrm{dR}} \longrightarrow 0. \tag{4}$$

Tensoring with V, we get a short exact sequence

$$0 \longrightarrow V \longrightarrow B_{\mathrm{cris}} \otimes V \longrightarrow (B_{\mathrm{cris}} \otimes V) \oplus (\widetilde{B}_{\mathrm{dR}} \otimes V) \longrightarrow 0$$

inducing a long exact sequence

$$0 \to H^0(K,V) \to D_{N=0} \to D_{N=0} \oplus t_V \longrightarrow H^1_f(K,V) \longrightarrow 0. \tag{S_f}$$

1.3. $H^1_{\mathrm{st}}(K,V)$

Let

$$B'_{\mathrm{st}} = \{(x,y) \in (B_{\mathrm{st}})^2 \mid p\varphi x - x = Ny\}.$$

If $z \in B_{\mathrm{st}}$, then $(Nz, \varphi z - z) \in B'_{\mathrm{st}}$. We denote $\iota : B_{\mathrm{st}} \to B'_{\mathrm{st}} \oplus \widetilde{B}_{\mathrm{dR}}$ the map $z \mapsto ((Nz, \varphi z - z), \tilde{z})$.

Lemma 1. *The sequence*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{\mathrm{st}} \stackrel{\iota}{\longrightarrow} B'_{\mathrm{st}} \oplus \widetilde{B}_{\mathrm{dR}} \longrightarrow 0 \tag{5}$$

is exact.

Proof. It is clear that $\ker(\iota) = B_{\mathrm{st}}^{N=0,\varphi=1} \cap B^+_{\mathrm{dR}} = \mathbb{Q}_p$. We only need to show that ι is surjective. Let $((x,y),w) \in B'_{\mathrm{st}} \oplus \widetilde{B}_{\mathrm{dR}}$. By surjectivity of $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$, there is a $z_1 \in B_{\mathrm{st}}$ such that $Nz_1 = x$. We have $N(y - (\varphi z_1 - z_1)) = p\varphi x - x - N(\varphi z_1 - z_1) = 0$, i.e. $y - (\varphi z_1 - z_1) \in B_{\mathrm{cris}}$. By surjectivity of $\varphi - 1 : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$, there is a $z_2 \in B_{\mathrm{cris}}$ such that $\varphi z_2 - z_2 = y - (\varphi z_1 - z_1)$. By surjectivity of $B_e \to \widetilde{B}_{\mathrm{dR}}$, there is a $z_3 \in B_e$ such that $\tilde{z}_3 = w - (\tilde{z}_1 + \tilde{z}_2)$. Let $z = z_1 + z_2 + z_3 \in B_{\mathrm{st}}$, then we have $\iota(z) = ((x,y),w)$. □

Tensoring (5) with V, we get a short exact sequence

$$0 \longrightarrow V \longrightarrow B_{\mathrm{st}} \otimes V \longrightarrow \big(B'_{\mathrm{st}} \otimes V\big) \oplus (\widetilde{B}_{\mathrm{dR}} \otimes V) \longrightarrow 0$$

inducing a long exact sequence

$$0 \to H^0(K,V) \to D \to D' \oplus t_V \longrightarrow H^1_{\mathrm{st}}(K,V) \longrightarrow 0 \tag{S_{st}}$$

where $D' = H^0(K, B'_{\mathrm{st}})$.

Moreover D' can be easily computed from D:

Proposition 2. *Denote $x \mapsto \bar{x}$ the projection of D onto D/ND and consider the maps*

$$\iota_0 : D_{N=0} \longrightarrow D \oplus D_{N=0}, \quad w \mapsto (w, -\varphi w + w),$$
$$\iota_1 : D \oplus D_{N=0} \to D \oplus D, \quad (u,v) \mapsto (Nu, \varphi u - u + v),$$
$$\iota_2 : D' \to D/ND, \quad (x,y) \mapsto \bar{x}.$$

The image of ι_1 *is contained in* D'*, the image of* ι_2 *is contained in* $(D/ND)_{\varphi=p^{-1}}$ *and the sequence*

$$0 \longrightarrow D_{N=0} \xrightarrow{\iota_0} D \oplus D_{N=0} \xrightarrow{\iota_1} D' \xrightarrow{\iota_2} (D/ND)_{\varphi=p^{-1}} \longrightarrow 0$$

is exact.

Proof. The inclusions

$$\mathrm{Image}(\iota_1) \subset D' \quad \text{and} \quad \mathrm{Image}(\iota_2) \subset (D/ND)_{\varphi=p^{-1}}$$

are obvious. We have:

$$D' = \{(x, y) \in D^2 \mid p\varphi x - x = Ny\}.$$

If $x \in D$ lifts $s \in (D/ND)_{\varphi=p^{-1}}$, then there exists $y \in D$ such that $Ny = p\varphi x - x$ and (x, y) is in D' and such that $\iota_2(x, y) = s$, hence ι_2 is onto.

If $(u, v) \in D \oplus D_{N=0}$, we have $\iota_2(\iota_1(u, v)) = \iota_2(Nu, \varphi u - u + v) = 0$. Conversely, if $(x, y) \in D'$ lies in the kernel of ι_2, it means there exists $u \in D$ such that $Nu = x$. Hence $(x, y) - \iota_1(u, 0)$ is an element of D' of the form $(0, v)$ and $Nv = 0$. Hence $(x, y) = \iota_1(u, v)$ and the image of ι_1 is the kernel of ι_2.

If $w \in D_{N=0}$, then $\iota_1(\iota_0(w)) = \iota_1(w, -\varphi w + w) = (Nw, \varphi w - w - \varphi w + w) = 0$. Conversely, if (u, v) lies in the kernel of ι_1, we have $Nu = 0$ and $v = -\varphi u + u$, hence $(u, v) = \iota_0(u)$.

The map ι_0 is obviously injective and it concludes the proof. □

The following result is now obvious:

Proposition 3. *The* $\mathbb{Q}_p$*-vector spaces* $H^1_f(K, V)/H^1_e(K, V)$ *and* $H^1_{\mathrm{st}}(K, V)/H^1_e(K, V)$ *are finite dimensional. We have:*

$$\dim_{\mathbb{Q}_p} H^1_f(K, V)/H^1_e(K, V) = \dim_{\mathbb{Q}_p} D_{N=0,\varphi=1}$$

and

$$\dim_{\mathbb{Q}_p} H^1_{\mathrm{st}}(K, V)/H^1_f(K, V) = \dim_{\mathbb{Q}_p} (D/ND)_{\varphi=p^{-1}}.$$

2. The case of a finite extension of $\mathbb{Q}_p$

We assume now that K is a finite extension of $\mathbb{Q}_p$. Recall that a p-adic Galois representation V of G_K is *potentially semi-stable* if there is a finite extension L of K contained in $\overline{K}$ such that, if L_0 is the fraction field of the ring of Witt vectors with coefficients in the residue field of L:

$$\dim_{\mathbb{Q}_p} V = \dim_{L_0} H^0(L, B_{\mathrm{st}} \otimes V).$$

In this case, we can use Proposition 3 to compute the dimension of $H^1_g(K, V)/H^1_f(K, V)$ and get Hyodo's celebrated result (cf. [6]):

Main Theorem. *For a potentially semi-stable representation* V*,*

$$H^1_g(K, V) = H^1_{\mathrm{st}}(K, V). \qquad (*)$$

The original proof of Hyodo, never published, used decomposition of iso-crystals and unramified representations. This result has been extended by Laurent Berger [1] to the general case (Berger proves that any de Rham representation is potentially semi-stable), but his proof is much more involved.

2.1. Reduction to the semi-stable case

We consider the commutative diagram

$$\begin{array}{ccccc}
H^1(K, V) & \xrightarrow{\alpha_K} & H^1(K, B_{\mathrm{st}} \otimes V) & \xrightarrow{\beta_K} & H^1(K, B_{\mathrm{dR}} \otimes V) \\
\downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\
H^1(L, V) & \xrightarrow{\alpha_L} & H^1(L, B_{\mathrm{st}} \otimes V) & \xrightarrow{\beta_L} & H^1(L, B_{\mathrm{dR}} \otimes V)
\end{array}$$

where L is a finite extension of K. The vertical arrows are injective by the relation $\mathrm{Cor} \circ \mathrm{Res} = [L : K]$. The above diagram shows that the injectivity of $\beta_L|_{\mathrm{Im}(\alpha_L)}$ implies the injectivity of $\beta_K|_{\mathrm{Im}(\alpha_K)}$.

By definition of $H^1_{\mathrm{st}}(K,V)$ and $H^1_g(K,V)$, we have the following commutative diagram, where the two horizontal sequences are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\mathrm{st}}(K,V) & \longrightarrow & H^1(K,V) & \xrightarrow{\alpha_K} & \mathrm{Im}(\alpha_K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\cong} & & \downarrow{\beta_K|_{\mathrm{Im}(\alpha_K)}} & & \\
0 & \longrightarrow & H^1_g(K,V) & \longrightarrow & H^1(K,V) & \longrightarrow & H^1(K, B_{\mathrm{dR}} \otimes V) & &
\end{array}$$

By the Snake Lemma, we know that $H^1_{\mathrm{st}}(K,V) = H^1_g(K,V)$ is equivalent to the injectivity of $\beta_K|_{\mathrm{Im}(\alpha_K)}$. So $(*)$ for K is equivalent to $(*)$ for L.

2.2. Computation of $\dim H^1_g(K,V)/H^1_f(K,V)$

Now assume V is semi-stable. Let $V^*(1)$ be the dual representation twisted by the Tate module of the multiplicative group. Recall the following result of Bloch and Kato [2, propo. 3.8]:

Lemma 4. *The usual perfect pairing of class field theory (given by the cup-product)*

$$H^1(K,V) \times H^1\big(K, V^*(1)\big) \longrightarrow H^2\big(K, \mathbb{Q}_p(1)\big) \xrightarrow{\sim} \mathbb{Q}_p,$$

is such that

(1) $H^1_e(K,V)$ and $H^1_g(K,V^*(1))$ are the exact annihilators of each other,
(2) $H^1_g(K,V)$ and $H^1_e(K,V^*(1))$ are the exact annihilators of each other,
(3) $H^1_f(K,V)$ and $H^1_f(K,V^*(1))$ are the exact annihilators of each other.

By the above Lemma, then

$$\dim_{\mathbb{Q}_p} H^1_g(K,V)/H^1_f(K,V) = \dim_{\mathbb{Q}_p} H^1_f\big(K,V^*(1)\big)/H^1_e\big(K,V^*(1)\big).$$

By Proposition 3, the latter one is equal to

$$\dim_{\mathbb{Q}_p} D_{\mathrm{st}}\big(V^*(1)\big)_{N=0,\varphi=1} = \dim_{\mathbb{Q}_p} D_{\mathrm{st}}\big(V^*\big)_{N=0,\varphi=p^{-1}}.$$

By duality, this is equal to

$$\dim_{\mathbb{Q}_p} \big((D/ND)^*\big)^{\varphi=p^{-1}} = \dim_{\mathbb{Q}_p} (D/ND)^{\varphi=p^{-1}},$$

which is equal to $\dim_{\mathbb{Q}_p} H^1_{\mathrm{st}}(K,V)/H^1_f(K,V)$ by using Proposition 3 again. This concludes the proof of the Main Theorem. □

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