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Abstract

Let U be a smooth affine curve over a number field K with a compactification X and let $\mathbb{L}$ be a rank 2, geometrically irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U with cyclotomic determinant that extends to an integral model, has Frobenius traces all in some fixed number field $E \subset \overline{\mathbb{Q}}_\ell$, and has bad, infinite reduction at some closed point x of $X \setminus U$. We show that $\mathbb{L}$ occurs as a summand of the cohomology of a family of abelian varieties over U. The argument follows the structure of the proof of a recent theorem of Snowden and Tsimerman, who show that when $E = \mathbb{Q}$, then $\mathbb{L}$ is isomorphic to the cohomology of an elliptic curve $E_U \to U$.

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1. Introduction

To state our main result, we require the following definition and setup.

Definition 1.1. Let B/k be a smooth variety over a finitely generated field and let $\ell \neq \mathrm{char}(k)$ be a prime. An abelian scheme $g\colon A_B \to B$ is said to be *of* SL_2-*type* if there is a decomposition of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on B:

$$R^1 g_* \overline{\mathbb{Q}}_\ell \cong \bigoplus_i \mathbb{L}_i^{m_i}, \tag{1.1}$$

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where each $\mathbb{L}_i$ is a geometrically irreducible rank 2 lisse $\overline{\mathbb{Q}}_\ell$-sheaf on B with cyclotomic determinant $\bigwedge^2 \mathbb{L}_i \cong \overline{\mathbb{Q}}_\ell(1)$.

Setup 1.2. Let K be a number field and let X/K be a smooth, proper, geometrically irreducible curve. Let $U \subset X$ be a Zariski open and dense subset of X with reduced complementary divisor D. Assume that D is non-empty.

Let $f\colon A_U \to U$ be a generically simple abelian scheme that is of SL_2-type and has bad, infinite reduction along some non-empty subset of D. Then the following statements hold for each direct summand $\mathbb{L}_i$ of $R^1 f_* \overline{\mathbb{Q}}_\ell$.

(1) The summand $\mathbb{L}_i$ is a geometrically irreducible, rank 2 lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U with cyclotomic determinant: $\bigwedge^2 \mathbb{L}_i \cong \overline{\mathbb{Q}}_\ell(1)$.
(2) There exists a proper smooth model $\mathfrak{X}$ over $\mathcal{O}_K[1/N]$, an open subset $\mathfrak{U}$ of $\mathfrak{X}$ extending U, an ℓ-adic local field M, and a lisse $\mathcal{O}_M$-sheaf $\mathcal{L}_i$ on $\mathfrak{U}$ such that

$$(\mathcal{L}_i \otimes_{\mathcal{O}_M} \overline{\mathbb{Q}}_\ell)|_U \cong \mathbb{L}_i.$$

(3) There exists a number field E such that for each closed point x of $\mathfrak{U}$, the trace of Frobenius on $(\mathcal{L}_i)_x$ is in $E \subset \overline{\mathbb{Q}}_\ell$.
(4) The local (geometric) monodromy of $\mathbb{L}_i$ is infinite around some non-empty subset of D.

In [ST18], Snowden and Tsimerman prove that when $E = \mathbb{Q}$, the above four conditions characterize those lisse $\overline{\mathbb{Q}}_\ell$-sheaves coming from families of elliptic curves. More precisely, they prove the following.

Theorem 1.3 (Snowden–Tsimerman). *Let the notation be as in Setup 1.2 and let $\mathbb{L}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U satisfying the above conditions (1)–(4), with $E = \mathbb{Q}$. Then there exists a family of elliptic curves*

$$f : A_U \to U$$

and an isomorphism $\mathbb{L} \cong \mathrm{R}^1 f_*(\overline{\mathbb{Q}}_\ell)$.

In this article, we consider the situation where Frobenius traces are all contained in a fixed number field E.

Theorem 1.4. *Let the notation be as in Setup 1.2 and let $\mathbb{L}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U satisfying conditions (1)–(4). Then there exists an abelian scheme*

$$f : A_U \to U$$

such that $\mathbb{L}$ is a summand $\mathrm{R}^1 f_*(\overline{\mathbb{Q}}_\ell)$.

Remark 1.5. We view Theorem 1.4 as providing a bit of further evidence for the relative Fontaine–Mazur conjecture, as in [LZ17, Conjecture, p. 292] or [Pet23, Conjecture 1].

An observation of Litt implies that for an arithmetic local system, condition (2) will automatically hold: see step 2 of the proof of [Lit21, Theorem 1.1.3] or [Pet23, Theorem 6.1]. (See also the argument in [LZ17, Proposition 4.1].) Therefore, to prove the relative Fontaine–Mazur conjecture for rank 2 local systems that have infinite monodromy around some point, it suffices to bound the field generated by Frobenius traces. This task seems to be quite difficult in general; for some progress on this question, see [Shi20].

Remark 1.6. We do not have any idea how to get around point (4). As will be explained in the proof sketch, this is because we crucially use some of Drinfeld's early work on the Langlands correspondence for GL_2 over function fields. More specifically, he is able to show that if $\mathbb{L}$ is

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an irreducible rank 2 lisse $\overline{\mathbb{Q}}_\ell$-sheaf over a curve $U/\mathbb{F}_q$ with cyclotomic determinant and infinite monodromy at ∞, then $\mathbb{L}$ comes from a family of abelian varieties over U. His proof finds such an abelian scheme as an isogeny factor of a Drinfeld modular curve over $\mathbb{F}_q(U)$.[1] When we do not assume infinite monodromy at ∞, then no such result is known; more specifically, the output of his later work on the Langlands correspondence will imply that there exists an open subset $V \subset U \times U$ and a smooth projective morphism $f\colon S \to V$ of *relative dimension 2* such that $\mathbb{L} \boxtimes \mathbb{L}^*|_V$ is a summand of $R^2 f_* \overline{\mathbb{Q}}_\ell$. See [Kri22, Remark 1.4, Question 9.1] and [KP21, § 1] for related discussion.

Our argument largely follows [ST18], but we need several new ingredients. To explain this, we quickly reprise their argument in the following remark.

Remark 1.7 (Sketch of [ST18]). For notational simplicity, assume that $\mathbb{L}$ corresponds to a representation

$$\rho\colon \pi_1(U_K) \to \mathrm{GL}_2(\mathbb{Z}_\ell),$$

with the property that the mod ℓ^3 residual representation $\pi_1(U_K) \to \mathrm{GL}_2(\mathbb{Z}/\ell^3\mathbb{Z})$ is trivial.

(1) Using Drinfeld's first work on the Langlands correspondence over finite fields, for all $\mathfrak{p} \gg 0$ they construct families of elliptic curves over $\mathfrak{U}_\mathfrak{p}$ with trivial ℓ^3 torsion whose monodromy is isomorphic to $\rho|_{\mathfrak{U}_\mathfrak{p}}$. (This involves an implicit isogeny from what is produced by Drinfeld's theorem.) These families, in turn, induce maps

$$\lambda_\mathfrak{p}\colon \mathfrak{X}_\mathfrak{p} \to \bar{\mathcal{M}}_{1,1}(\ell^3) \to \bar{\mathcal{M}}_{1,1}(\ell^3) \otimes \mathcal{O}_K/\mathfrak{p},$$

where $\bar{\mathcal{M}}_{1,1}(\ell^3)$ is the compactified moduli space of elliptic curve with full ℓ^3 level structure, defined over $\mathrm{Spec}(\mathbb{Z}[1/\ell])$, and the final target is therefore a hyperbolic curve over $\mathcal{O}_K/\mathfrak{p}$.

(2) While the map $\lambda_\mathfrak{p}$ is not *a priori* generically separable, they factor it through absolute Frobenius to construct a new map, $\mu_\mathfrak{p}\colon \mathfrak{X}_\mathfrak{p} \to \bar{\mathcal{M}}_{1,1}(\ell^3) \otimes \mathcal{O}_K/\mathfrak{p}$ which is generically separable. Note that the induced elliptic curve over $\mathfrak{U}_\mathfrak{p}$ also has monodromy isomorphic to $\rho|_{\mathfrak{U}_\mathfrak{p}}$. Then Riemann–Hurwitz applies, bounding the degree of the map $\mu_\mathfrak{p}$ by some number d, which is crucially independent of $\mathfrak{p}$. We may replace $\lambda_\mathfrak{p}$ with $\mu_\mathfrak{p}$.

(3) At this point, consider the moduli space of maps:

$$\mathcal{H} := \mathrm{Hom}^{\leq d}_{\mathcal{O}_K[1/N]}(\mathfrak{X}, \bar{\mathcal{M}}_{1,1}(\ell^3)),$$

of morphisms of curves λ over $\mathcal{O}_K[1/N]$, with degree bounded by d. This moduli space is a scheme of finite type over $\mathcal{O}_K[1/N]$ because we have put a bound on the degree.[2] For each k, let $\mathcal{H}_k$ denote the subset of $\mathcal{H}$ consisting of those maps λ such that:

- $\lambda(\mathfrak{U}) \subset \mathcal{M}_{1,1}(\ell^3)$; and
- the induced elliptic curve $E_U \to U$ has mod ℓ^k monodromy isomorphic to ρ mod ℓ^k.

Then Snowden and Tsimerman show that each the $\mathcal{H}_k$ is a closed subset and, hence, so is $\mathcal{H}_\infty := \cap \mathcal{H}_k$. The subset $\mathcal{H}_\infty \subset \mathcal{H}$ will parametrize those maps λ such that the monodromy representation is *integrally* isomorphic to ρ. Equipping $\mathcal{H}_\infty \subset \mathcal{H}$ with the reduced induced subscheme structure, they deduce that $\mathcal{H}_\infty$ is therefore a scheme of finite type over $\mathcal{O}_K[1/N]$. As it has points

[1] One crucial technique he uses is an equal-characteristic p rigid-analytic uniformization of the Drinfeld modular curve at a cusp; indeed, the Drinfeld modular curve will be totally degenerating.

[2] In fact, potentially at the cost of increasing N, this moduli space is *finite flat* over $\mathcal{O}_K[1/N]$. However, this fact is not used in their proof.

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modulo $\mathfrak{p}$ for infinitely many primes $\mathfrak{p}$ of $\mathcal{O}_K$, it follows that it has a point over a finite extension field K'/K. Then a Weil restriction argument, together with Faltings' isogeny theorem, allows one to conclude.

We now explain the new ingredients in turn, highlighting the additional difficulties.

Remark 1.8 (Sketch of the proof of Theorem 1.4). Again for notational simplicity, assume that $\mathcal{L}$ corresponds to a representation:

$$\rho\colon \pi_1(U_K) \to \mathrm{GL}_2(\mathbb{Z}_\ell).$$

(Note that $\mathbb{Q}_\ell$ contains number fields of infinitely large degree.) We further assume that ρ has the property that the mod ℓ^3 residual representation $\pi_1(U_K) \to \mathrm{GL}_2(\mathbb{Z}/\ell^3\mathbb{Z})$ is trivial. (This last assumption will play no role, but we include it to see which additional technicalities emerge.)

(1) Again using Drinfeld's early work on the Langlands correspondence over finite fields, for each $\mathfrak{p} \gg 0$, we may construct an abelian scheme over $f_\mathfrak{p}\colon A_\mathfrak{p} \to \mathfrak{U}_\mathfrak{p}$ of dimension $h := [E : \mathbb{Q}]$, such that $\mathcal{L}|_{\mathfrak{U}_\mathfrak{p}}$ injects as a summand of $\mathrm{R}^1 f_{\mathfrak{p},*}\overline{\mathbb{Q}}_\ell$.

(a) Here we encounter our first complication: it is *not necessarily true* that we can choose $A_\mathfrak{p}[\ell^3]$ to be the split étale cover of $\mathfrak{U}_\mathfrak{p}$: unlike in the case [ST18], $\mathcal{L}|_{\mathfrak{U}_\mathfrak{p}}$ is not all of $\mathrm{R}^1 f_{\mathfrak{p},*}\overline{\mathbb{Q}}_\ell$. However, there exists a *finite, connected* cover $\varphi_\mathfrak{p}\colon (\mathfrak{X}_\mathfrak{p})' \to \mathfrak{X}_\mathfrak{p}$ (purely in characteristic p!) of degree $\leq |\mathrm{GL}_{2h}(\mathbb{Z}/\ell^3\mathbb{Z})|$ such that:
   - the map $\varphi_\mathfrak{p}$ is finite étale over $\mathfrak{U}_\mathfrak{p}$;
   - if we set $(\mathfrak{U}_\mathfrak{p})' := \varphi_\mathfrak{p}^{-1}(\mathfrak{U}_\mathfrak{p})$, the pullback $A' \to (\mathfrak{U}_\mathfrak{p})'$ has trivial ℓ^3-torsion;
   - the abelian scheme $f'_\mathfrak{p}\colon A' \to (\mathfrak{U}_\mathfrak{p})'$ has semistable reduction at $(\mathfrak{X}_\mathfrak{p})'$.

   The key property of the cover $\varphi\colon (\mathfrak{X}_\mathfrak{p})' \to \mathfrak{X}_\mathfrak{p}$ is that the degree is bounded independent of $\mathfrak{p}$. However, we emphasize that, as of yet, there is no preferred $X' \to X$ over K that patches all of these modulo $\mathfrak{p}$ covers together. At this point, we demand that $N > |\mathrm{GL}_{2h}(\mathbb{Z}/\ell^3\mathbb{Z})|$, to ensure that any such $\varphi_\mathfrak{p}$ is tamely ramified.

(b) We now encounter our next (minor) trouble. *A priori*, there is no bound on the degree of the polarization of $f'_\mathfrak{p}\colon A'_\mathfrak{p} \to (\mathfrak{U}_\mathfrak{p})'$. This has a simple solution: Zarhin's trick, which says that $B'_\mathfrak{p} := (A'_\mathfrak{p} \times (A'_\mathfrak{p})^t)^4$ has a principal polarization.

(c) There is a third trouble; unlike in the approach of Snowden and Tsimerman, we have not yet nailed down the integral monodromy, and this is more subtle. There are several ways one could address this. Our solution to this problem will be found in the construction of a simple moduli space, $\mathcal{H}_k$: see step (3).

We therefore get a map:

$$\lambda'_\mathfrak{p}\colon (\mathfrak{X}_\mathfrak{p})' \to \mathcal{A}^*_{8h,1,\ell^3} \to \mathcal{A}^*_{8h,1,\ell^3} \otimes \mathcal{O}_K/\mathfrak{p},$$

where $\mathcal{A}^*_{8h,1,\ell^3}$ is the Baily–Borel compactification of the fine moduli scheme $\mathcal{A}_{8h,1,\ell^3}$ parametrizing principally polarized abelian schemes of dimension $8h$ and trivial level ℓ^3 structure. This $\lambda'_\mathfrak{p}$ has the following property: the pullback of the universal rank $16h$ lisse ℓ-adic sheaf on $\mathcal{A}_{8h,1,\ell^3}$ to $\mathfrak{U}_\mathfrak{p}$ has ρ as a *rational* summand.

(2) Our next goal is to somehow numerically bound $\lambda'_\mathfrak{p}$. Recall that [ST18] do this by a combination of Riemann–Hurwitz and factoring through some power of absolute Frobenius. In our setting, this step is more tricky, and we chose to use an Arakelov-style inequality. More precisely, if $f_\mathfrak{p} : \bar{B}'_\mathfrak{p} \to (\mathfrak{X}_\mathfrak{p})'$ is the Néron model of $B'_\mathfrak{p} \to \mathfrak{U}'_\mathfrak{p}$, then we will bound the degree of

the Hodge vector bundle $E^{1,0}_{(\mathfrak{X}_\mathfrak{p})'} := R^0 f_{\mathfrak{p}*}\Omega^1_{\bar{B}'_\mathfrak{p}/(\mathfrak{X}_\mathfrak{p})'}(\log \Delta)$, at least for many infinitely many $\mathfrak{p}$. Set $E^{0,1}_{(\mathfrak{X}_\mathfrak{p})'} := R^1 f_{p*}\mathcal{O}_{\bar{B}'_\mathfrak{p}}$. Then to bound the degree of $E^{1,0}_{(\mathfrak{X}_\mathfrak{p})'}$, we will need to know that *the logarithmic Kodaira–Spencer map* constructed by Faltings and Chai,

$$\theta'_\mathfrak{p} : E^{1,0}_{(\mathfrak{X}_\mathfrak{p})'} \to E^{0,1}_{(\mathfrak{X}_\mathfrak{p})'} \otimes \Omega^1_{(\mathfrak{X}_\mathfrak{p})'}(\log D),$$

is not only non-zero but is moreover an isomorphism at the generic point.[3] In more detail: for any $\mathfrak{p} \gg 0$ such that the underlying prime number p splits completely in E, the field generated by Frobenius traces, the induced p-divisible group on $(\mathfrak{U}_\mathfrak{p})'$ splits as the direct sum of several copies of h (mutually non-isogenous) height 2, dimension 1 p-divisible groups G'_i and their duals $(G'_i)^t$. We prove, using monodromy considerations, that they are generically ordinary and have supersingular points. Applying a Frobenius untwisting lemma from the PhD thesis of Jie Xia [Xia13], we conclude that we may 'Frobenius untwist' each of them until they are all generically versally deformed.[4] (In Appendix A, we provide a proof of Xia's Frobenius untwisting lemma in our context, and also give a second argument and perspective of the termination of Frobenius untwisting stability techniques.) Once again using Zarhin's trick, we will obtain an isogenous, principally polarized abelian scheme over $\mathfrak{U}'_\mathfrak{p}$, which we relabel $B'_\mathfrak{p}$, with the Néron model

$$f_\mathfrak{p} : \bar{B}'_\mathfrak{p} \to (\mathfrak{X}_\mathfrak{p})'$$

and such that the logarithmic Kodaira–Spencer map is a generically injective map of coherent sheaves on $(\mathfrak{X}_\mathfrak{p})'$. By taking determinants, we deduce an Arakelov-style inequality, thereby bounding the degree of the induced Hodge line bundle on $(\mathfrak{X}_\mathfrak{p})'$ by some integer d, which is crucially *independent of* $\mathfrak{p}$. The output of this is Lemma 2.7.

(3) To mimic the third step, we first construct some finite-type moduli spaces of $\mathcal{O}_K[1/N]$, and then we use our argument as above to show it has points modulo $\mathfrak{p}$ for infinitely many $\mathfrak{p}$. This is, in greater detail, as follows.

(a) Fix $d > 1$ and set $\mathcal{H}$ to be the moduli of triples $(\mathfrak{X}', \varphi, \lambda)$:

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{\lambda} & \mathcal{A}^*_{8h,1,\ell^3} \otimes \mathcal{O}_K[1/N] \\ \downarrow{\scriptstyle \varphi} & & \\ \mathfrak{X}, & & \end{array}$$

where:
- $\mathfrak{X}'/\mathcal{O}_K[1/N]$ is a smooth, proper, geometrically connected curve;
- φ is finite, of degree at most $\leq |\mathrm{GL}_{16h}(\mathbb{Z}/\ell^3\mathbb{Z})|$, and étale over $\mathfrak{U}$;
- there exists some point $\infty' \in \mathfrak{X}'$ that is sent to a 0-dimensional cusp in $\mathscr{A}^*_{8h,1,\ell^3}$; and
- the degree of the pulled-back Hodge line bundle on $\mathfrak{X}'_K$ is $\leq d$.

Then $\mathcal{H}$ will be a Deligne–Mumford stack, of finite type over $\mathcal{O}_K[1/N]$. (The stackiness comes from the intervening Hurwitz space.) We further show that the generic fiber of $\mathcal{H}$

[3] The logarithmic Kodaira–Spencer map θ'_p is the derivative of the period map $\lambda'_\mathfrak{p}$. We emphasize that knowing that map $\lambda'_\mathfrak{p}$ is generically separable (equivalently, $\theta'_\mathfrak{p}$ being non-zero) is not sufficient for our application. Morally speaking, we need to prove that the image of $\lambda'_\mathfrak{p}$ is not contained in one of the natural foliations of the relevant Hilbert modular variety. However, we chose to not work with good reductions of Hilbert modular varieties, as that theory is more complicated than the bare theory of moduli of abelian varieties.

[4] This mimics the elliptic modular setting for the following reason: the equal-characteristic universal deformation space of a height 2, dimension 1 p-divisible group is one-dimensional.

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has dimension 0 (and is, in fact, reduced); while this is an immediate corollary of the PhD thesis of Ben Moonen [Moo98], we argue following the work of Saito. This implies that, after potentially further increasing N, the stack $\mathcal{H}/\mathcal{O}_K[1/N]$ has relative dimension 0.[5]

(b) Recall that $\mathcal{L}$ is a lisse $\mathbb{Z}_\ell$-sheaf on $\mathfrak{U}$, whose generic fiber is an $\mathbb{Z}_\ell$-lattice inside of $\mathbb{L}$. For $k \geq 1$, set $\mathcal{H}_k$ to be the subspace of $\mathcal{H}$ given by those $(\mathfrak{X}', \varphi, \lambda)$ (with induced abelian scheme $f\colon B' \to \mathfrak{X}'$) such that there exists a map

$$\psi\colon \varphi^*(\mathcal{L})/\ell^k \to R^1 f_* Z_\ell/\ell^k$$

of torsion locally constant étale sheaves with the following condition: the reduction modulo ℓ of ψ is non-zero. (This condition is crucial in our approach.)[6] Then $\mathcal{H}_k \subset \mathcal{H}$ will be a closed substack, which we may equip with the reduced induced structure. Similarly, set $\mathcal{H}_\infty$ to be $\cap \mathcal{H}_k$, again with the reduced induced stack structure. Note that $\mathcal{H}_\infty$ is then a finite-type Deligne–Mumford stack over $\mathrm{Spec}(\mathcal{O}_K[1/N])$ for some N.

Unlike in the Snowden–Tsimerman approach, the relationship of the moduli space $\mathcal{H}_\infty$ to Drinfeld's theorem is not immediately apparent. However, in both approaches, the moduli spaces involve extra maps of *ℓ^k-torsion sheaves* rather than lisse $\mathbb{Z}_\ell$-sheaves.

(c) By the careful choice of $\mathcal{H}_k$ and a crucial diagonalization argument on $\mathcal{H}_\infty$ (contained in Lemma 3.6), it will follow from the earlier steps that there exists an infinite set of primes $\mathfrak{p}$ such that $\mathcal{H}_\infty$ has points modulo $\mathfrak{p}$. (Unlike the approach of Snowden and Tsimerman, this does not require one to take an ℓ-primary isogeny.) By the Nullstellensatz, one deduces that $\mathcal{H}_\infty$ has characteristic 0 points. A Weil restriction argument then yields the result.

Remark 1.9. Katz has shown that rigid local systems on the punctured projective line are motivic, and Corlette and Simpson have shown that all rigid rank 2 local systems are motivic. Our main theorem provides a new arithmetic approach to both Katz's theorem in rank 2 and also the Corlette–Simpson theorem, subject to an additional assumption analogous to condition (4). Here is an outline of the proof. We emphasize that these approaches will critically rely on a quasi-projective version of a deep theorem on projective varieties of Esnault and Groechenig [EG20]; this result was announced very recently [EG23].

First we assume that U is a curve. Let $\mathbb{L}$ be a cohomologically rigid local system of rank 2 on $U^{an}_{\mathbb{C}}$ with coefficients in $\overline{\mathbb{Q}}_\ell$, trivial determinant and infinite monodromy around ∞. Suppose that the local system $\mathbb{L}$ spreads out to an étale local system $\mathcal{L}$ with cyclotomic determinant over a finitely generated spreading out $\mathfrak{U}/S$ such that the *stable Frobenius trace fields* are bounded, i.e. there exists a number field E such that for all closed points s, there exists a finite extension s'/s such that the Frobenius trace field of $\mathcal{L}|_{\mathfrak{U}_{s'}}$ is contained in E. Then, our argument applies verbatim to prove that $\mathbb{L}$ over $U_{\mathbb{C}}$ comes from a family of abelian varieties; we get mod p points for infinitely many $p \gg 0$, and the relevant moduli space is of finite type and, in fact, generically 0-dimensional, so by specialization of the prime-to-p fundamental group we may conclude.

In fact, recent work [KL23] of the first-named author and Lam shows the following. If $X/\mathbb{C}$ is a projective variety, and if $\mathbb{L}$ is a cohomologically rigid $\overline{\mathbb{Q}}_\ell$-local system with trivial determinant on X^{an}, then there exists a spreading out $\mathfrak{X}/S$ and a number field $E \subset \overline{\mathbb{Q}}_\ell$ such that $\mathbb{L}$ spreads out to an étale local system $\mathcal{L}$ on $\mathfrak{X}$ with cyclotomic determinant such that the stable

[5] By specialization of the prime-to-p fundamental group, the fact that $\mathcal{H}$ has relative dimension 0 over $\mathcal{O}_K[1/N]$ will imply that the geometric local system $\mathbb{L}|_{U_{\overline{K}}}$ comes from a family of abelian varieties. This does not use any of the more delicate moduli spaces $\mathcal{H}_k$ to come.

[6] When the lisse $\overline{\mathbb{Q}}_\ell$-sheaf has coefficients in an ℓ-adic local field M, not necessarily assumed to be $\mathbb{Q}_\ell$, then we instead demand that there exists such a ψ whose reduction modulo a fixed uniformizer π_M is non-zero.

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Frobenius trace field is contained inside of E. In the quasi-projective case, which is what is needed here, the arguments of [KL23] work *mutatis mutandis*, substituting in [EG23] for [EG20]. Indeed, in [KL23], the key p-adic fact we need is that rigid stable flat connections give rise to F^f-isocrystals on the relevant p-adic completions.

In general, when U is higher dimensional (i.e. $U = X \setminus D$, where X is a smooth projective variety and D is a simple normal crossings divisor), it is very plausible that one may similarly deduce the analog of the Corlette–Simpson theorem here (again, subject to the restriction that the local system is cohomologically rigid and that the local monodromy around one of the boundary divisors is infinite). Here is a sketch of the argument. The main results of [EG23] in fact output rank 2 *filtered logarithmic F-crystals*; as above, porting these objects into [KL23] as above, one can deduce that cohomologically rigid rank 2 local systems will have spreading-outs whose stable Frobenius trace field is bounded. A complete set of companions to the logarithmic F-isocrystals so constructed will likely exist, as in the projective case this is shown in [EG20]. From these F-isocrystals, [KP22] will provide abelian schemes on open subsets of the mod p fibers of bounded dimension[7] and [KP21, Corollary 6.12] shows that, after possibly replacing with an isogenous abelian scheme, the abelian schemes extend to the whole mod p fiber of $\mathfrak{U}$. We can bound the degree of the Hodge line bundle for infinitely many p by Frobenius untwisting, exactly as is done here. Finally, the appropriate Hom scheme will again be 0-dimensional, so by using specialization of the prime-to-p fundamental group one may again conclude.

2. Drinfeld's work on the Langlands correspondence for GL_2 and some corollaries

A key ingredient in the proof of Theorem 1.4 is the following Theorem 2.2, which is a byproduct of Drinfeld's first work on the Langlands correspondence for GL_2. We first record a setup.

Setup 2.1. Let p be a prime number and let $q = p^a$. Let $C/\mathbb{F}_q$ be a smooth, affine, geometrically irreducible curve with smooth compactification $\bar{C}$. Let $Z := \bar{C} \setminus C$ be the reduced complementary divisor.

Theorem 2.2 (Drinfeld). *Let the notation be as in Setup 2.1 and let $\mathbb{L}$ be a rank 2 irreducible $\overline{\mathbb{Q}}_\ell$ sheaf on C with determinant $\overline{\mathbb{Q}}_\ell(1)$. Suppose $\mathbb{L}$ has infinite local monodromy around some point at $\infty \in Z$. Then $\mathbb{L}$ comes from a family of abelian varieties in the following sense: let E be the field generated by the Frobenius traces of $\mathbb{L}$ and suppose $[E : \mathbb{Q}] = h$. Then there exists an abelian scheme*

$$\pi_C \colon A_C \to C$$

of dimension h and an isomorphism $E \cong \mathrm{End}_C(A) \otimes \mathbb{Q}$, realizing A_C as a SL_2-type abelian scheme, such that $\mathbb{L}$ occurs as a summand of $R^1(\pi_C)_*\overline{\mathbb{Q}}_\ell$. Moreover, $A_C \to C$ is totally degenerate around ∞.

See [ST18, Proof of Proposition 19, Remark 20] for how to recover this result from Drinfeld's work. This amounts to combining [Dri83, Main Theorem, Remark 5] with [Dri77, Theorem 1].

We make some observations about the p-adic properties of the resulting abelian schemes. In particular, our goal is to show that, in the context of Theorem 2.2, we can modify $A_C \to C$ with products, duals, and isogenies such that the resulting abelian scheme $B_C \to C$ that has especially nice (p-adic) properties; these will, in turn, allow us to prove an

[7] The boundedness of the dimension comes from the boundedness of the stable Frobenius trace field.

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Arakelov-style inequality. First, we will give the following non-standard definition, which is adapted for our purpose.

Definition 2.3. Maintain notation as in Setup 2.1. Let G_C be a p-divisible group on C. We say G_C has *strong strict semistable reduction along* Z if:

- G_C has semistable reduction along Z (see [Tri08, Definition 4.2]), which is based on semistable reduction in the sense of de Jong [dJ98, Definition 2.2]; and
- if for every point $\infty \in \bar{C}\backslash C$ with local parameter z_∞ the restricted p-divisible group
$$G_C|_{\mathrm{Spec}(\mathbb{F}_q((z_\infty)))}$$
over $\mathrm{Spec}(\mathbb{F}_q((z_\infty)))$ does not extend to a p-divisible group over $\mathrm{Spec}(\mathbb{F}_q[[z_\infty]])$.

Definition 2.3 is useful as it concisely expresses the condition that G_C have semistable reduction and, moreover, that it does not extend as a p-divisible group across *any* of the cusps.

The next proposition will be critical for bounding degrees of maps to moduli spaces. In Appendix A, we explain a second proof/perspective of the second part, which is based on a destabilizing iteration argument due to Langer.

Proposition 2.4. *Maintain notation as in Setup 2.1. Let G_C be a height 2, dimension 1 p-divisible group on C with strong strict semistable reduction along Z. Suppose further that $\mathbb{D}(G_C) \otimes \overline{\mathbb{Q}}_p$ is an irreducible object of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(C)_{\overline{\mathbb{Q}}_p}$. Then:*

1. *G_C is generically ordinary with a non-empty supersingular locus; and*
2. *there exists an isogenous p-divisible group $H_C \to G_C$ that is generically versally deformed (in the sense of [Kri22, Defintions 8.1, 8.2]).*[8]

Before we begin the proof, we comment on the overconvergence assumption. If $H_C \to C$ is a p-divisible group, then F-isocrystal $\mathbb{D}(H_C)$ is automatically a convergent F-isocrystal. In our setting, the fact that we demand $G_C \to C$ to be semistable around Z implies that $\mathbb{D}(G_C)$ is, in fact, overconvergent. Part of the hypothesis of Proposition 2.4 is then that $\mathbb{D}(G_C) \otimes \mathbb{Q}_p$ is absolutely irreducible in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(C)$.

First proof of Proposition 2.4. As G_C has height 2 and dimension 1, there are only two possible Newton polygons, which correspond to the p-divisible group being ordinary or supersingular, respectively. If G_C were not generically ordinary, it would be everywhere supersingular. However, supersingular p-divisible groups cannot be strictly semistable: as there is no multiplicative part, the filtration in [dJ98, Definition 2.2], would have to be trivial, which would imply that G_C extends to a p-divisible group over $\bar{C}$. This shows G_C is generically ordinary.

Suppose that G_C had no supersingular points: then G_C is everywhere ordinary. Let H_C be the multiplicative sub-p-divisible group of G_C, i.e. the height 1, dimension 1 p-divisible group with Newton slope 1 everywhere. Let $\infty \in Z$, with formal parameter z_∞. Then the p-divisible group $G_C|_{\mathrm{Spec}(\mathbb{F}_q((z_\infty)))}$ has semistable reduction in the sense of [dJ98, Definition 2.2] and does not extend to a p-divisible group over $\mathrm{Spec}(\mathbb{F}_q[[z_\infty]])$. Then, for the definition of semistability to

[8] We briefly recall the notion here. Let $G_C \to C$ be a height 2, dimension 1 p-divisible group. There is a Kodaira–Spencer map: $KS\colon T_C \to \Omega^* \otimes \Psi$, where Ω is the Hodge line bundle of G_C and Ψ is the dual of the Hodge line bundle of the Serre dual G_C^t. We say that $G_C \to C$ is *generically versally deformed* if the above KS is non-zero. After the work of Illusie [Ill85], this is equivalent to the following condition: there exists a closed point c such that the map $u_c\colon C_c^\wedge \to \mathrm{Def}(G_c)$ from the formal completion of C at c to the equal-characteristic universal deformation space of G_c is a formally smooth map of formally smooth, 1-dimensional $\kappa(c)$ schemes, i.e. u_c is an isomorphism.

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be satisfied, the only possible filtration is

$$G^{\mu}_{\mathrm{Spec}(\mathbb{F}_q((z_\infty)))} = G^{f}_{\mathrm{Spec}(\mathbb{F}_q((z_\infty)))} = H_C|_{\mathrm{Spec}(\mathbb{F}_q((z_\infty)))}.$$

(Here, the meaning of $G^{\mu}_{\mathrm{Spec}(\mathbb{F}_q((z_\infty)))}$ and $G^{f}_{\mathrm{Spec}(\mathbb{F}_q((z_\infty)))}$ is given as in [dJ98, Definition 2.2].)

However, by the definition of semistability, $H_C|_{\mathrm{Spec}(\mathbb{F}_q((z_\infty)))}$ therefore extends to a p-divisible group over $\mathrm{Spec}(\mathbb{F}_q[[z_\infty]])$. Ranging over all points $\infty \in Z$, we see that $\mathbb{D}(H_C) \otimes \mathbb{Q}_p \in \mathbf{F\text{-}Isoc}(C)$ in fact extends to an F-isocrystal on $\mathbf{F\text{-}Isoc}(\bar{C})$: therefore, $\mathbb{D}(H_C) \otimes \mathbb{Q}_p \in \mathbf{F\text{-}Isoc}^{\dagger}(C)$. However, this yields a sub-object (in $\mathbf{F\text{-}Isoc}^{\dagger}(C)$) of $\mathbb{D}(G_C) \otimes \mathbb{Q}_p$, contradicting the absolute irreducibility of the hypothesis. Therefore, G_C has a non-empty supersingular locus.

Now, suppose that $G_C \to C$ is not generically versally deformed, i.e. that $KS = 0$ identically on C. Then by [Xia13, Theorem 6.1], there is a p-divisible group $(G_1)_C \to C$ such that $(G_1)^{(p)}_C \cong G_C$, i.e. the Frobenius twist of $(G_1)_C$ is isomorphic to G_C. The p-divisible groups G_C and $(G_1)_C$ are isogenous. If the Kodaira–Spencer map for $(G_1)_C$ is non-zero, we may stop. Otherwise, we may apply [Xia13, Theorem 6.1] again to find a p-divisible group $(G_2)_C \to C$ such that $(G_2)^{(p)}_C \cong (G_1)_C$. We claim this process must terminate at some point. Here is a simple proof (also indicated in [Kri22, p. 253]). Let c be a closed point of C such that G_c is supersingular. Then the (equal characteristic) deformation map:

$$u_c \colon C^{\wedge}_{/c} \to \mathrm{Def}(G_c) \cong \kappa(c)[[t]]$$

is non-zero, because G_C is generically ordinary. (In other words, if u_c were 0, then the p-divisible group over $\mathrm{Spec}(\kappa(c)[[z_c]])$ would be supersingular at both the closed and the generic point, which is a contradiction: over the generic point, the p-divisible group is base-changed from $\mathbb{F}_q(C)$ along the map $\mathbb{F}_q(C) \hookrightarrow \kappa(c)[[z_c]]$.)

The map KS_c is simply the derivative of u_c. In particular, $KS_c = 0$ implies that $u_c^*(t) \in \kappa(c)[[z_c]]$ is a power series in z_c^p; on the level of the universal deformation map u_c, Frobenius untwisting amounts to extracting a pth root of $u_c^*(t)$. As u_c is not constant, this process must terminate. ∎

Corollary 2.5. *Let the notation be as in Theorem 2.2. Further, suppose the following.*

- *The lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathbb{L}$ has infinite, unipotent local monodromy around each point $\infty \in Z$.*
- *Let E be the field generated by Frobenius traces of $\mathbb{L}$. Suppose that p splits completely in E.*

Then there exists an abelian scheme $f_C \colon A_C \to C$ satisfying all of the conclusions of Drinfeld's theorem 2.2, together with the further properties

$$A_C[p^\infty] \cong \bigoplus_i G_{C,i},$$

where:

1. *the $G_{C,i}$ are all mutually non-isogenous;*
2. *each $G_{C,i}$ is a height 2, dimension 1 p-divisible group on C; and*
3. *each $G_{C,i}$ is generically versally deformed, generically ordinary, and has non-empty supersingular locus.*

Proof. We will first construct $\bigoplus_i G_{C,i}$ with the desired properties.

Let $f \colon A_C \to C$ be an abelian scheme produced by Drinfeld's theorem 2.2. By Grothendieck's monodromy criterion for semistable reduction, $A_C \to C$ is totally degenerate around every point of Z. The F-isocrystal $\mathcal{E} := \mathbb{D}(A_C[p^\infty]) \otimes \mathbb{Q}_p$ is a semi-simple object of $\mathbf{F\text{-}Isoc}^{\dagger}(C)$ by [Pál22]. We claim that $\mathcal{E}$ is the companion to $R^1(\pi_C)_*\overline{\mathbb{Q}}_\ell$. Indeed, a theorem of Zarhin

[Mor85, Chapitre XII, Theorem 2.5, pp. 244–245] implies that $R^1(\pi_C)_*\overline{\mathbb{Q}}_\ell$ is semi-simple and the characteristic polynomials of Frobenius agree at closed points by [KM74]. On the other hand, there is a decomposition:

$$\mathcal{E}_{\overline{\mathbb{Q}}_p} \cong \bigoplus_i \mathcal{E}_i, \tag{2.1}$$

where $\mathcal{E}_i$ are *irreducible* objects of $\mathbf{F\text{-}Isoc}^\dagger(C)_{\overline{\mathbb{Q}}_p}$.

It follows from [KP22, Remark 2.8] that every summand $\mathcal{E}_i$ is a companion of $\mathbb{L}$.[9] As the relation of companions preserves 'infinite monodromy at $\infty \in Z$', each $\mathcal{E}_i$ has infinite monodromy around every $\infty \in Z$.

In addition, $\det(\mathcal{E}_i) = \overline{\mathbb{Q}}_p(1)$, again because the property 'cyclotomic determinant' is preserved under the companions relation.

As p splits completely in E, it follows that $E \otimes \mathbb{Q}_p \cong \Pi\mathbb{Q}_p$ acts on $\mathcal{E}$, and the images of the idempotents are the $\mathcal{E}_i$, i.e. the (absolutely) irreducible summands $\mathcal{E}_i$ are objects of $\mathbf{F\text{-}Isoc}^\dagger(C)$.

The slopes of each $\mathcal{E}_i$ are in between 0 and 1. Therefore, we may apply [KP21, Lemma 5.8][10] and [dJ95] to see that for each $\mathcal{E}_i$, there exists a (non-canonical) p-divisible group $G_{C,i}$ with $\mathbb{D}(G_{C,i}) \otimes \overline{\mathbb{Q}}_p \cong \mathcal{E}_i$. (Equivalently, note that $\mathcal{E}_i \in \mathbf{F\text{-}Isoc}^\dagger(C)$, i.e. each $\mathcal{E}_i$ has coefficients in $\mathbb{Q}_p$, by the hypothesis that p splits completely in E.)

The p-divisible groups $A_C[p^\infty]$ and $\bigoplus_i G_{C,i}$ are isogenous. At this point, we wish to claim that each $G_{C,i}$ has strong strict semistable reduction along Z. First of all, note that $A_C[p^\infty]$ has strong strict semistable reduction by [dJ98, 2.5].

As $\mathbb{D}(G_{C,i}) \otimes \mathbb{Q}_p$ is overconvergent, it follows from [Pál22, Theorem 2.22] that every $G_{C,i}$ has semistable reduction along Z. Suppose for contradiction that $G_{C,1}$ extended through some cusp $\infty \in Z$. Then $\mathcal{E}_1 \cong \mathbb{D}(G_{C,1}) \otimes \mathbb{Q}_p$ also extends to an (overconvergent) F-isocrystal on the curve $C \cup \{\infty\} = \bar{C} \setminus \{Z \setminus \infty\}$. As each of the $\mathcal{E}_i$ are companions, this implies that they all also extend to $C \cup \{\infty\}$. Therefore, the ℓ-adic companion also extends to $C \cup \{\infty\}$. This implies that $\mathbb{L}$ also extends to a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $C \cup \{\infty\}$, contradicting our assumption that $\mathbb{L}$ had infinite, unipotent monodromy around ∞.

We may now apply Proposition 2.4 to replace each $G_{C,i}$ with an isogenous p-divisible group that satisfies the two conclusions of the proposition. Note that we still have the relation $A_C[p^\infty]$ is isogenous to $\bigoplus G_{C,i}$.

By [KP22, Lemma 2.13], it follows that we can replace A_C by an isogenous abelian scheme such that

$$A_C[p^\infty] \cong \bigoplus_i G_{C,i},$$

where every $G_{C,i}$ is generically versally deformed, is generically ordinary, and has supersingular points. Finally, each G_i will be mutually non-isogenous because the F-isocrystals $\mathbb{D}(G_i) \otimes \mathbb{Q}_p$ are a complete collection of p-adic companions of $\mathbb{L}$ (see [KP22, Remark 2.8]). □

Using the above, we will be able to extract all of the p-adic information we need from Theorem 2.2 to prove Theorem 1.4. We need one final piece of notation.

Definition 2.6. Let $N \geq 1$ be an integer prime to p and let $g \geq 1$ be a positive integer. Then $\mathscr{A}_{g,1,N}/\mathrm{Spec}(\mathbb{Z}[1/N])$ denotes the (fine) moduli space of principally polarized abelian varieties

[9] Strictly speaking, the remark only states this for the ℓ-adic companion, but the preceding sentence then follows immediately from the theory of companions.
[10] This is really due to Katz, see [Kat79, Theorem 2.6.1].

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with trivial full level N structure. This is a smooth, quasi-projective scheme over $\mathrm{Spec}(\mathbb{Z}[1/N])$. It has a compactification, $\mathscr{A}^*_{g,1,N}/\mathrm{Spec}(\mathbb{Z}[1/N])$.[11] This latter scheme has a natural ample line bundle, the *Hodge line bundle*, which we denote by α.

Then the precise output we need from Drinfeld's theorem 2.2 is given in the following lemma.

Lemma 2.7. *Let the notation be as in Theorem 2.2. Suppose that p splits completely in E and $\mathbb{L}$ has infinite, unipotent monodromy around every point of Z.*

Then there exists a principally polarized abelian scheme abelian scheme $f\colon B_C \to C$, of SL_2 type and dimension $8h$, such that $\mathbb{L}$ occurs as a direct summand of $R^1 f_* \overline{\mathbb{Q}}_\ell$, and the following hold.

(1) *The abelian scheme $B_C \to C$ has semistable, infinite reduction along $\bar{C}\backslash C$. Call the Néron model $\bar{B}_{\bar{C}} \to \bar{C}$.*

(2) *There exist h mutually non-isogenous p-divisible groups $G_{C,i}$, each of height 2, dimension 1, and generically versally deformed, such that there is a decomposition of p-divisible groups*

$$B_C[p^\infty] \cong \bigoplus_i (G_{C,i} \times G^t_{C,i})^4.$$

(3) *After Kato and Trihan, to $\bar{f}\colon \bar{B}_{\bar{C}} \to \bar{C}$ there is an associated logarithmic F-crystal with nilpotent residues (M, F) in finite, locally free modules on the log pair $(\bar{C}, Z)$. Similarly, there is a* logarithmic Hodge vector bundle, *which we write as $\Omega_{\bar{B}/\bar{C}}$, a rank $8h$ vector bundle on $\bar{C}$. Then the following hold:*
   (i) *$\Omega_{\bar{B}/\bar{C}}$ splits as the direct sum of $8h$ positive line bundles on $\bar{C}$;*
   (ii) *the log Kodaira–Spencer map (constructed in [FC90, Ch. III, Corollary 9.8]*

$$\theta\colon \Omega_{\bar{B}/\bar{C}} \to R^1 \bar{f}_*(\mathcal{O}_{\bar{B}}) \otimes \omega_{\bar{C}}(Z),$$

   where $\omega_{\bar{C}}$ denotes the sheaf of differential one-forms on $\bar{C}$, is an injective map of coherent sheaves on $\bar{C}$;
   (iii) $\deg(\Omega_{\bar{B}/\bar{C}}) \leq h/2 \cdot (2g(\bar{C}) - 2 + |Z|) = 4h\chi_{\mathrm{top}}(C)$;
   (iv) *suppose N is an integer, coprime to p, such that $B_C[N] \to C$ is a split étale cover; then the induced moduli map $C \to \mathscr{A}_{8h,1,N}$ extends to a map*

$$\bar{C} \to \mathscr{A}^*_{8h,1,N},$$

   where the latter denotes the Baily–Borel *compactification; then the Hodge line bundle α on $\mathscr{A}^*_{8h,1,N}$ pulls back to* $\det(\Omega_{\bar{B}/\bar{C}})$.

Proof. Construct $A_C \to C$ as in Corollary 2.5. Again, by Grothendieck's criterion for semistable reduction of abelian varieties, $A_C \to C$ must have semistable reduction. Set $B_C := (A_C \times A^t_C)^4$; then by a result of Zarhin [Mor81, Chapitre IX, Lemme 1.1, p. 205], $B_C \to C$ is principally polarized. Moreover, it clearly has semistable reduction. From the construction, and the fact that $A^t_C[p^\infty] \cong (A_C[p^\infty])^t$, where the first transpose is 'dual abelian scheme' and the second is 'Serre-dual p-divisible group', it follows that part (2) holds.

We are left to prove part (3). To do this, we will make heavy use of [KP22, Setup A.10, Proposition A.11]. First of all, each $\mathbb{D}(G_{C,i})$, *a priori* an Dieudonné crystal on C, extends uniquely to logarithmic Dieudonné crystal (with nilpotent residues) on $(\bar{C}, Z)$. Indeed, existence of the extension of $\mathbb{D}(B_C[p^\infty])$ follows from [KT03, (4.4)–(4.8)] and uniqueness from [KP22,

[11] The moduli space $\mathscr{A}_{g,1,N}$ is not geometrically connected over $\mathrm{Spec}(\mathbb{Z}[1/N])$. This is why some authors prefer to work with the geometrically connected components, which are defined over $\mathrm{Spec}(\mathbb{Z}[\zeta_N, 1/N])$.

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Proposition A.11(3)]: name the extension (M, F, V). These two results immediately imply the desired existence and uniqueness for the extension of $\mathbb{D}(G_i)$ to a logarithmic F-crystal, which we name (M_i, F, V). The uniqueness implies that the (unique) extension of $\mathbb{D}(G_i^t)$ is isomorphic to the *dual* logarithmic Dieudnné crystal $(M_i, F, V)^t$ by [KP21, (5.11) and (5.12)].

Set $M_{(\bar{C},Z)}$ to be the *evaluation* of M on the trivial thickening of $(\bar{C}, Z)$ and set Ω to be the kernel of F on $M_{(\bar{C},Z)}$; then Ω is a vector bundle on $\bar{C}$, called the *Hodge vector bundle*. (Kato and Trihan obtain the dual version of this in [KT03, (5.1)], especially Lemma 5.3 of [KT03].)[12] Similarly, we can construct the Hodge bundle Ω_i of each (M_i, F), which will be a *line bundle* on $\bar{C}$. Moreover, there is a short exact sequence

$$0 \to \Omega_i \to (M_i)_{(\bar{C},Z)} \to \Psi_i^* \to 0,$$

where Ψ_i is the Hodge bundle of G_i^t. We have an isomorphism of vector bundles on $\bar{C}$:

$$\Omega \cong \bigoplus_i (\Omega_i \times \Psi_i)^4$$

As each $G_{C,i}$ has non-empty supersingular locus, it follows that the *Hasse invariant* associated to $G_{C,i}$,

$$\mathrm{Hasse}_{G_{C,i}} \in H^0(\bar{C}, \Omega_i^{\otimes p-1}),$$

is non-zero, which implies that Ω_i is a positive degree line bundle on $\bar{C}$. As $G_{C,i}^t$ is supersingular exactly when $G_{C,i}$ is supersingular, we deduce that Ψ_i is also positive. Therefore, $\Omega = \Omega_{\bar{B}/\bar{C}}$ splits as the direct sum of $8h$ positive line bundles. In particular, we have shown part (i). We further note that Ω is isomorphic to the Hodge line bundle associated to the Néron model of $B_C \to C$ by [KP22, (A.11)] (this was first proven in [KT03]).

For the next step, Faltings and Chai have constructed the following Kodaira–Spencer map [FC90, Ch. III, Corollary 9.8]:

$$\Omega_{\bar{B}/\bar{C}} \otimes \Omega_{\bar{B}^t/\bar{C}} \to \omega_{\bar{C}}(Z), \tag{2.2}$$

extending the usual Kodaira–Spencer map over C. As B admits a principal polarization, we have that $B_C \cong B_C^t$ and, hence, $\bar{B}_{\bar{C}} \cong \bar{B}^t{}_{\bar{C}}$, as both are simply the respective Néron models. Therefore, we may equivalently write (2.2) as

$$\theta\colon \Omega_{\bar{B}/\bar{C}} \to \Omega^*_{\bar{B}/\bar{C}} \otimes \omega_{\bar{C}}(Z) \cong \mathcal{H}om(\Omega_{\bar{B}/\bar{C}}, \omega_{\bar{C}}(Z)).$$

Under the decomposition

$$\Omega_{\bar{B}/\bar{C}} \cong \bigoplus (\Omega_i \oplus \Psi_i)^4,$$

and after restricting to C, the above $\theta|_C$ is just the sum of the Kodaira–Spencer maps for each G_i and G_i^t:

$$\begin{aligned} \Omega_i|_C &\to \mathcal{H}om_{\mathcal{O}_C}((\Psi_i)|_C, \omega_C), \\ \Psi_i|_C &\to \mathcal{H}om_{\mathcal{O}_C}((\Omega_i)|_C, \omega_C). \end{aligned}$$

These were constructed to be non-zero, as both G_i and G_i^t are generically versally deformed; therefore, the Kodaira–Spencer map of sheaves is an injective map of coherent sheaves. Therefore, part (ii) is shown.

[12] See also [KP22, Remarks A.8 and A.9] for a discussion of these results and references for the compatibility with the usual Hodge vector bundle associated to an abelian scheme.

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Fortunately, part (iii) is an easy corollary of part (ii). Indeed, taking degrees, we see that

$$\deg(\Omega_{\bar{B}/\bar{C}}) \leq \deg(\mathcal{H}om(\Omega_{\bar{B}/\bar{C}}, \omega_{\bar{C}}(Z))) = 2g(\bar{C}) - 2 + |Z| - \deg(\Omega_{\bar{B}/\bar{C}}),$$

from which the inequality follows immediately.

Finally, let us prove part (iv). By moduli theory, we have a map $C \to \mathscr{A}_{8h,1,N}$, where the latter is a fine moduli scheme. As $\bar{C}$ is a smooth curve and the Baily–Borel compactification $\mathscr{A}^*_{8h,1,N}$ is proper, it follows that we get an extension:

$$\lambda\colon \bar{C} \to \mathscr{A}^*_{8h,1,N}.$$

Finally, the argument that the Hodge line bundle on $\mathscr{A}^*_{8h,1,N}$ pulls back under λ to $\det(\Omega_{\bar{B}/\bar{C}})$ is given in the text surrounding [KP22, (3.4), (3.5)]. (While the argument in [KP22] is only written for $N = \ell$, it generalizes verbatim to the matter at hand. Indeed, the argument is an easy corollary of [FC90, Ch. V, Theorem 2.5].) □

3. The moduli spaces

We work in the following situation. Let K be number field, let $N \geq 1$, set $S := \mathrm{Spec}(\mathcal{O}_K[1/N])$, and let $\mathfrak{X}/S$ be a smooth projective curve, let $\mathfrak{D} \subset \mathfrak{X}$ be a relative reduced divisor, and let $\mathfrak{U}$ denote the open complement.

Let ℓ be a prime number and let $g \geq 1$ be an integer. We again denote by $\mathscr{A}^*_{g,1,\ell^3}$ the Baily–Borel compactification of $\mathscr{A}_{g,1,\ell^3}$, which is defined over $\mathrm{Spec}(\mathbb{Z}[1/\ell])$. This moduli space has a natural ample line bundle, the Hodge line bundle, which we denote by α.

Definition 3.1. Fix a positive integer b. Denote by $\mathcal{H}$ the following contravariant pseudo-functor from the category of S schemes to the 2-category of groupoids. The value $\mathcal{H}(T)$ for an S-scheme T is the groupoid of triples $(\mathfrak{Y}, \varphi, \lambda)$, that fit into a diagram

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\ \lambda\ } & \mathscr{A}^*_{g,1,\ell^3} \times T \\ {\scriptstyle\varphi}\downarrow & & \\ \mathfrak{X}_T & & \end{array}$$

where

- $\mathfrak{Y}/T$ is a smooth, projective, geometrically connected curve;
- λ sends $\mathfrak{W} := \varphi^{-1}(\mathfrak{U}) \subset \mathfrak{Y}$ to $\mathscr{A}_{g,1,\ell^3} \times T \subset \mathscr{A}^*_{g,1,\ell^3} \times T$;
- φ is finite morphism, of degree $\leq |\mathrm{GL}_{2g}(\mathbb{Z}/\ell^3\mathbb{Z})|$, and étale over $\mathfrak{U}$;
- there exists some cusp ∞ of $\mathfrak{Y}(T)$ (lying over a point of $\mathfrak{D}(T)$ of $\mathfrak{X}(T)$) that is sent to a 0-dimensional cusp of $\mathscr{A}^*_{g,1,\ell^3} \times T$; and
- the degree of the pulled-back Hodge line bundle $\lambda^*(\alpha)$ on every geometric fiber of $\mathfrak{Y}$ is $\leq b$;

with the natural notion of isomorphism that if $(\mathfrak{Y}, \varphi, \lambda)$ and $(\mathfrak{Y}', \varphi', \lambda')$ are elements of $\mathcal{H}(T)$, then an isomorphism between them is an $\mathfrak{X}_T$-isomorphism $\mathfrak{Y} \to \mathfrak{Y}'$ that intertwines φ and φ' as well as λ and λ'.

Colloquially, the functor $\mathcal{H}$ parameterizes finite covers $\mathfrak{Y}$ of $\mathfrak{X}$ equipped with a (principally polarized) abelian scheme of dimension g with trivial ℓ^3 level structure, such that the induced map $\mathfrak{Y} \to \mathscr{A}^*_{g,1,\ell^3}$ has 'degree' bounded by b.

Proposition 3.2. *After potentially increasing N (equivalently, replacing S by a non-empty open subscheme), the functor $\mathcal{H}$ is represented by a finite-type Deligne–Mumford stack over S.*

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Proof. By increasing N, we can assume that all covers $\varphi\colon \mathfrak{Y} \to \mathfrak{X}$ that occur in the definition of $\mathcal{H}$ are tamely ramified at the cusps. More precisely, this will hold for any $N > |\mathrm{GL}_{2g}(\mathbb{Z}/\ell^3\mathbb{Z})|$.

It follows from the theory of the Hilbert scheme that for any noetherian scheme T and for any relative smooth proper curve $\mathfrak{Y}/T$ with geometrically connected fibers, the functor $\mathrm{Hom}_T^{\leq b}(\mathfrak{Y}, \mathscr{A}^*_{g,1,\ell^3})$ parametrizing maps λ such that the $\deg(\lambda^*(\alpha)) \leq b$ for every geometric fiber is represented by a finite-type T-scheme. (In particular, this holds true even if T is not connected and the genus of the fibers varies on different connected components.) It follows that if T is a Deligne–Mumford stack, of finite type over S, and $\mathfrak{Y}/T$ is a smooth, proper curve with geometrically connected fibers, then the same functor $\mathrm{Hom}_T^{\leq b}(\mathfrak{Y}, \mathscr{A}^*_{g,1,\ell^3})$ is represented by a finite-type Deligne–Mumford stack over T.

On the other hand, the theory of the Hurwitz scheme implies that the functor $\mathrm{Cov}^c_{(\mathfrak{X},\mathfrak{D})/S}$ parametrizing finite tame covers $\mathfrak{Y} \to \mathfrak{X}$ of degree $\leq c$ such that:

- $\mathfrak{Y}/S$ has geometrically connected fibers; and
- $\mathfrak{Y} \to \mathfrak{X}$ is étale over $\mathfrak{U} := \mathfrak{X} \setminus \mathfrak{D}$

is represented by a finite-type Deligne–Mumford stack over S.

There is a natural map

$$\mathrm{Cov}^{\leq c}_{(\mathfrak{X},\mathfrak{D})/S} \to \mathscr{M} := \bigsqcup_{k \text{ bounded}} \mathscr{M}_k,$$

which is the map that sends a cover $\mathfrak{Y} \to \mathfrak{X}$ to the underlying curve $\mathfrak{Y}$. Here, the notation $\mathscr{M}_k$ stands for the moduli space of genus k curves. Denote by $\mathscr{C}_k \to \mathscr{M}_k$ the universal curve and by $\mathscr{C} := \bigsqcup \mathscr{C}_k$, which has a natural map $\mathscr{C} \to \mathscr{M}$.

Then consider $\mathscr{H}$, the open substack of the 2-fiber product:

$$\mathrm{Hom}^{\leq b}_{\mathscr{M}}(\mathscr{C}, \mathscr{A}^*_{g,1,\ell^3})) \times_{\mathscr{M}} \mathrm{Cov}^{\leq |\mathrm{GL}_{2g}(\mathbb{Z}/\ell^3\mathbb{Z})|}_{(\mathfrak{X},\mathfrak{D})/S},$$

which corresponds to the condition that sends $\lambda(\varphi^{-1}(\mathfrak{U})) \subset \mathscr{A}_{g,1,\ell^3}$, i.e. that $\varphi^{-1}(\mathfrak{U})$ is sent inside of the moduli space of abelian varieties. It follows that $\mathscr{H}$ is finite-type Deligne–Mumford stack over S. By further imposing the condition that the map λ sends at least one point in the boundary divisor to a zero-dimensional cusp of the Baily–Borel compactification, $\mathcal{H}$ is a closed substack of $\mathscr{H}$, which is again finite type. □

Now, let $\mathbb{L}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf as in Theorem 1.4. There exists an ℓ-adic local field $M/\mathbb{Q}_\ell$ such that the associated representation factors through the ring of integers $\mathcal{O}_M$:

$$\rho\colon \pi_1(\mathfrak{X}_{\mathcal{O}_K[1/N]}) \to \mathrm{GL}_2(\mathcal{O}_M) \subset \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).$$

Abusing notation, we call the induced lisse $\mathcal{O}_M$-sheaf $\mathcal{L}$. Denote by π_M the uniformizer of M and κ_M the residue field of M.

DEFINITION 3.3. Fix $i \geq 1$ and a lattice $\mathcal{L}$ as above. Let $\tilde{\mathcal{H}}_i$ denote the following contravariant pseudo-functor from S-schemes to groupoids: the value $\tilde{\mathcal{H}}_i(T)$ on an S-scheme T is the collection of quadruples $(\mathfrak{Y}, \varphi, \lambda, \psi)$, where $(\mathfrak{Y}, \varphi, \lambda) \in \mathcal{H}(T)$, and ψ is the following extra piece of data. As $\lambda\colon \mathfrak{W} := \varphi^{-1}(\mathfrak{U}) \to \mathscr{A}_{g,1,\ell^3}$, there is a principally polarized abelian scheme $f\colon A_{\mathfrak{W}} \to \mathfrak{W}$ (with trivial ℓ^3-torsion). Then

$$\psi\colon \varphi^*(\mathcal{L}/\pi_M^i) \to R^1 f_* \mathcal{O}_M/\pi_M^i$$

is a map of étale torsion sheaves on $\mathfrak{W}$ whose reduction modulo π_M-reduction is non-zero. In other words, $\mathrm{im}(\psi) \not\subseteq \pi_M(R^1 f_* \mathcal{O}_M/\pi_M^i)$. There is an obvious notion of isomorphism of two such quadruples.

The pseudo-functor $\tilde{\mathcal{H}}_i$ is actually a stack in the étale topology. This follows from the following two properties. Let T be a scheme.

- There exists an internal Hom in the category of torsion locally constant abelian étale sheaves on T.
- If $\psi\colon \mathcal{F} \to \mathcal{G}$ is a map of torsion, locally constant étale sheaves of $\mathcal{O}_M$ modules on T, then the property that $\psi(\mathcal{F}) \not\subset \pi_M(\mathcal{G})$ may be checked on an étale cover.

There are natural transformations of pseudo-functors $\tilde{\mathcal{H}}_j \to \tilde{\mathcal{H}}_i$ for any $j > i$. We claim that $\tilde{\mathcal{H}}_i$ represents a finite-type Deligne–Mumford stack over S. To prove this, it suffices to prove that the natural transformation of pseudo-functors $\tilde{\mathcal{H}}_i \to \mathcal{H}$ is representable by a scheme.

Let T be an S-scheme, and $t := (\mathfrak{Y}, \varphi, \lambda) \in \mathcal{H}(T)$. Then we have the following pullback square.

$$\begin{array}{ccc} T \times_{\mathcal{H}} \tilde{\mathcal{H}}_i & \longrightarrow & \tilde{\mathcal{H}}_i \\ \downarrow & & \downarrow \\ T & \xrightarrow{t} & \mathcal{H} \end{array}$$

Then $T \times_{\mathcal{H}} \tilde{\mathcal{H}}_i$ has the following description. There are two natural ℓ^i-torsion étale sheaves on $\mathfrak{Y}$: $\varphi^*(\mathcal{L}/\ell^i)$ (which has $\mathcal{O}_M/\ell^i$-rank 2), and $R^1 f_* \mathcal{O}_M/\ell^i$ (which has $\mathcal{O}_M/\ell^i$-rank $2g$). Then $T \times_{\mathcal{H}} \tilde{\mathcal{H}}_i$ corresponds to the (finite) set of injective maps of sheaves of abelian groups: $\psi\colon \varphi^*(\mathcal{L}/\ell^i) \hookrightarrow R^1 f_* \mathcal{O}_M/\ell^i$. This finite set is canonically a scheme. It follows that $\tilde{\mathcal{H}}_i \to \mathcal{H}$ is relatively representable, and hence $\tilde{\mathcal{H}}_i$ is represented by a Deligne–Mumford stack of finite type over S.

Proposition 3.4. *The natural map 'forget ψ': $\tilde{\mathcal{H}}_i \to \mathcal{H}$ is finite.*

Proof. It is obviously quasi-finite because, as argued above, if we fix i, then there are only finitely many choices for ψ. To prove it is finite, we show that it is proper.

As both $\tilde{\mathcal{H}}_i$ and $\mathcal{H}$ are of finite type over $S = \mathrm{Spec}(\mathcal{O}_K[1/N])$, it suffices to simply check the valuative criterion for properness. Let R be a discrete valuation ring with fraction field F. Suppose we have $(\mathfrak{Y}, \varphi, \lambda) \in \mathcal{H}(R)$ and $(\mathfrak{Y}_F, \varphi_F, \lambda_F, \psi_F) \in \tilde{\mathcal{H}}_i(F)$. Therefore, we have a principally polarized abelian scheme $f'\colon A_{\mathfrak{W}} \to \mathfrak{W}$ (of dimension g, with trivial ℓ^3 torsion), together with a map of torsion étale sheaves over $\mathfrak{W}_F$ whose reduction modulo π_M is non-trivial:

$$\psi_F\colon \varphi^*(\mathcal{L})/\pi_M^i|_{\mathfrak{W}_F} \to R^1 f'_* \mathcal{O}_M/\pi_M^i|_{\mathfrak{W}_F}.$$

Note the following. If one has two finite étale sheaves on an irreducible normal scheme, and a morphism between them over the generic point, then that morphism uniquely extends to the whole scheme. (Here, we are closely following [ST18, Proof of Lemma 23].) Therefore, ψ_F extends to a ψ on all of $\mathfrak{U}'_R$, and we have verified the valuative criterion for properness. □

Definition 3.5. For $i \geq 1$, set

$$\mathcal{H}_i := \mathrm{Im}(\tilde{\mathcal{H}}_i \to \mathcal{H}).$$

As $\tilde{\mathcal{H}}_i \to \mathcal{H}$ is relatively representable and finite (Proposition 3.4), it is universally closed. Therefore, $\mathcal{H}_i$ is a closed subset of $|\mathcal{H}|$, which we may equip with the induced reduced substack structure [Sta22, Tag 0508]. According the natural transformations $\tilde{\mathcal{H}}_j \to \tilde{\mathcal{H}}_i$, the sequence of

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closed subsets are descending. Set

$$\mathcal{H}_\infty := \bigcap_j \mathcal{H}_j,$$

which is also equipped with the reduced induced substack structure. Then $\mathcal{H}_i$ and $\mathcal{H}_\infty$ are Deligne–Mumford stacks of finite type over $S = \mathrm{Spec}(\mathcal{O}_K[1/N])$ for all $i \geq 1$.

Lemma 3.6. *Let T be an S-scheme and let $(\mathfrak{Y}, \varphi, \lambda) \in \mathcal{H}(T)$. Then the following conditions are equivalent:*

(1) $(\mathfrak{Y}, \varphi, \lambda) \in \mathcal{H}_\infty(T)$;
(2) *there exists an injection*

$$\varphi^*(\mathcal{L}) \hookrightarrow R^1 f_* \mathcal{O}_M$$

of lisse $\mathcal{O}_M$-sheaves on T;
(3) *there is an injection $\tau\colon \varphi^*(\mathcal{L} \otimes M) \hookrightarrow R^1 f_* M$ of lisse M-sheaves on T.*

Proof. (2) ⇒ (3) By applying $- \otimes_{\mathcal{O}_M} M$ to the injection $\varphi^*(\mathcal{L}) \hookrightarrow R^1 f_* \mathcal{O}_M$, we get the desired injection. (Note that both are lisse $\mathcal{O}_M$-sheaves, so tensoring with M yields an injective map.)

(3) ⇒ (2) Since $R^1 f_* \mathcal{O}_M$ (respectively, $\mathcal{L}$) is an $\mathcal{O}_M$-lattice in $R^1 f_* M$ (respectively, $\mathcal{L} \otimes M$), there exists some integer ι such that

$$\pi_M^\iota \cdot \tau\big(\varphi^*(\mathcal{L})\big) \subset R^1 f_* \mathcal{O}_M.$$

Then the map $\pi_M^\iota \tau$ is an injection from $\varphi^*(\mathcal{L})$ to $R^1 f_* \mathcal{O}_M$.

(2) ⇒ (1) Denote by ψ' the injection in condition (2). It is clear there exists some integer λ such that

$$\mathrm{im}(\psi') \subset \pi_M^\lambda R^1 f_* \mathcal{O}_M \quad \text{and} \quad \mathrm{im}(\psi') \not\subset \pi_M^{\lambda+1} R^1 f_* \mathcal{O}_M.$$

Denote $\psi = \psi'/\pi_M^\lambda$ which is clearly an injection from $\varphi^*(\mathcal{L})$ to $R^1 f_* \mathcal{O}_M$ and satisfies

$$\mathrm{im}(\psi) \not\subset \pi_M R^1 f_* \mathcal{O}_M.$$

This is equivalent to saying that the reduction modulo π_M of ψ is nontrivial. Denote $\psi_i = \psi \bmod (\pi_M^i)$ for each $i > 0$,

$$\psi_i\colon \varphi^*(\mathcal{L})/\pi_M^i \to R^1 f_* \mathcal{O}_M/\pi_M^i.$$

Since $\psi_i \bmod (\pi_M) = \psi \bmod (\pi_M) \neq 0$, the quadruple $(\mathfrak{Y}, \varphi, \lambda, \psi_i) \in \tilde{\mathcal{H}}_i$. Thus, $(\mathfrak{Y}, \varphi, \lambda) \in \cap_{i=1}^\infty \mathcal{H}_i = \mathcal{H}_\infty$.

(1) ⇒ (3) (This is the main content of the lemma.) Since $(\mathfrak{Y}, \varphi, \lambda) \in \mathcal{H}_\infty(T)$, for each $i > 0$, there exists a map

$$\psi_i'\colon \varphi^*(\mathcal{L})/\pi_M^i \to R^1 f_* \mathcal{O}_M/\pi_M^i$$

which is non-trivial modulo π_M. In general, the ψ_i''s *do not* form a compatible sequence, i.e. it is possible that there exists $j > i$ with the following property: $\psi_j' \bmod (\pi_M^i) \not\equiv \psi_i'$. Therefore, one cannot directly take projective limits to find our desired map $\varphi^*(\mathcal{L}) \to R^1 f_* \mathcal{O}_M$. However, we claim we may derive a compatible sequence from ψ_i' as follows.

Consider the subset in the finite set

$$\Sigma_1 = \mathrm{Hom}(\varphi^*(\mathcal{L})/\pi_M, R^1 f_* \mathcal{O}_M/\pi_M)$$

consisting of all modulo π_M reductions of ψ_i':

$$\{\psi_i' \bmod (\pi_M) \mid i \geq 1\}.$$

By the pigeonhole principle, there exists a non-trivial map $\psi_1 \in \Sigma_1$ and an infinite subset $\mathbb{N}_1 \subset \mathbb{N}$ such that $\psi_i' \bmod (\pi_M) = \psi_1$ for any $i \in \mathbb{N}_1$.

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Suppose we have constructed a compatible sequence $\psi_1, \psi_2, \ldots, \psi_r$ and an infinite subset $\mathbb{N}_r \subset \mathbb{N}$ satisfying

$$\psi'_i \bmod (\pi_M^j) = \psi_j \in \mathrm{Hom}(\varphi^*(\mathcal{L})/\pi_M^j, R^1 f_* \mathcal{O}_M/\pi_M^j)$$

for any $i \in \mathbb{N}_r$ and $j \in \{1, 2, \ldots, r\}$. Then we consider the subset in the finite set

$$\Sigma_{r+1} = \{\rho \in \mathrm{Hom}(\varphi^*(\mathcal{L})/\pi_M^{r+1}, R^1 f_* \mathcal{O}_M/\pi_M^{r+1}) \mid \rho \bmod \pi_M^r = \psi_r\}$$

consisting of all modulo π_M^{r+1} reductions $\psi'_i \bmod (\pi_M^{r+1})$:

$$\{\psi'_i \bmod (\pi_M^{r+1}) \mid i \in \mathbb{N}_r\}.$$

Again by the pigeonhole principle, there exists a non-trivial map $\psi_{r+1} \in \Sigma_{r+1}$ and an infinite subset $\mathbb{N}_{r+1} \subset \mathbb{N}_r$ such that $\psi'_i \bmod (\pi_M^{r+1}) = \psi_{r+1}$ for any $i \in \mathbb{N}_{r+1}$.

Iteratively, we find a sequence $\psi_1, \psi_2, \ldots$ satisfying

$$\psi_j \bmod (\pi_M^i) = \psi_i$$

for each $j > i$. Taking projective limits and tensoring with M, one gets a non-zero map

$$\psi \colon \varphi^*(\mathcal{L} \otimes M) \to R^1 f_* M.$$

Since $\mathcal{L} \otimes M$ is irreducible, ψ is injective. □

4. Rigidity

In this section, we prove the following. Recall that $S = \mathrm{Spec}(\mathcal{O}_K[1/N])$.

Lemma 4.1. *Let $\mathcal{H}/S$ be as in § 3. Then, after potentially increasing N (equivalently, replacing S by a non-empty Zariski open subset), the relative dimension of $\mathcal{H}/S$ is* 0.

Proof. We have shown that $\mathcal{H}/S$ is a finite-type Deligne–Mumford stack. To show the desired result, it suffices to show that if $K \hookrightarrow \mathbb{C}$ is an embedding, then $\mathcal{H}_{\mathbb{C}}$ has dimension 0. Equivalently, we want to show that if $A_{U_{\mathbb{C}}} \to U_{\mathbb{C}}$ is a principally polarized abelian scheme that is totally degenerate at at least one cusp, then it is rigid. This immediately follows from Theorem 8.6 together with Lemma 3.4 and the following text of [Sai93]. □

5. The proof

Proof of Theorem 1.4. First, assume that $\mathbb{L}$ has bad, unipotent reduction around every cusp. Let $\mathcal{T}_1$ be the set of those prime $\mathfrak{p}$ of $\mathcal{O}_K$ with the following properties: the underlying prime p splits completely in E, and $p > \max(N, \ell^3)$. This is an infinite set by the Cebotarev density theorem. Let $\mathcal{L}_{\mathfrak{p}}$ be the restriction of $\mathcal{L}$ to $U_{\mathfrak{p}}$. Then $\mathcal{L}_{\mathfrak{p}}$ is irreducible by exactly the same argument as that of the first paragraph of [ST18, Proof of Lemma 24, p. 2053].

There are only finitely may subfields of E. It follows from the pigeonhole principle that there exists a subfield $F \subset E$ such that there exists infinitely many primes $\mathfrak{p} \in \mathcal{T}_1$ such that $\mathcal{L}_{\mathfrak{p}}$ has Frobenius traces in $F \subset E$. Call the collection of such primes $\mathcal{T}_2 \subset \mathcal{T}_1$. Let $\mathcal{H}$ and $\mathcal{H}_\infty$ be the moduli spaces from § 3 with $g = 8[F : \mathbb{Q}]$ and $b = 4h\chi_{\mathrm{top}}(U)$. Note that, after increasing N, both spaces have relative dimension 0 over $\mathcal{O}_K[1/N]$ by § 4.

First of all, note that for each $\mathfrak{p} \in \mathcal{T}_2$, $\mathcal{H}_\infty(\kappa(\mathfrak{p})) \neq \emptyset$. This follows by Lemma 2.7, especially part (3)(iv), together with Lemma 3.6. In more detail: Lemma 2.7 implies that we can find an abelian scheme $B_{\mathfrak{U}_{\mathfrak{p}}} \to \mathfrak{U}_{\mathfrak{p}}$ such that $\mathcal{L}_{\mathfrak{p}}^4$ injects in the cohomology, that has semistable reduction at infinity, and such that the Hodge bundle on $\mathfrak{X}_{\mathfrak{p}}$ has bounded degree. Lemma 3.6 then implies that such an abelian scheme corresponds to a point $\beta_{\mathfrak{p}}$ in $\mathcal{H}_\infty(\kappa(\mathfrak{p})) \subset \mathcal{H}(\kappa(\mathfrak{p}))$. Since $\mathcal{T}_2$ is infinite and $\mathcal{H}_\infty/S$ is of finite type, it follows that there exists a finite field extension K'/K and a point $\beta \in \mathcal{H}(K')$. In fact, as $\mathcal{H}_\infty$ has relative dimension 0, our point β may be chosen to be compatible with infinitely many of the $\beta_{\mathfrak{p}}$, where compatibility is defined in the obvious sense. By definition of $\mathcal{H}_\infty$, the point $\beta \in \mathcal{H}_\infty(K')$ corresponds to an abelian scheme $B_{U'_{K'}} \to U'_{K'}$ such that $\mathcal{L}|_{U'_{K'}}$ injects into the integral $\mathcal{O}_M$ cohomology. By taking a Weil restriction, we obtain an abelian scheme $A_U \to U$ (of dimension $g[K':K]$) such that $\mathbb{L}$ injects into the cohomology of $A_U \to U$. Using Faltings' semi-simplicity theorem, we conclude that $\mathbb{L}$ is, in fact, a summand of the cohomology, as desired.

In general, there exists a finite étale cover $f\colon U' \to U$ such that $f^*\mathbb{L}$ has the following property. Let C' be the compactification of U', and set D' to be the divisor at infinity. Then for each $\infty \in D'$, the lisse ℓ-adic sheaf $f^*\mathbb{L}$ has either good reduction at ∞ or bad, unipotent reduction at ∞. There then exists a curve $U' \subset V' \subset C'$, where $f^*\mathbb{L}$ extends to a lisse ℓ-adic sheaf $\mathbb{M}'$ on all of V' and, moreover, has bad, unipotent reduction around every point in $C' \setminus V'$. Then the above argument applies, producing an abelian scheme $A_{V'} \to V'$ whose cohomology has $\mathbb{M}'$ as a summand. Restricting to U' and then applying a Weil restriction of scalars along the finite étale map U'/U, we obtain the desired result. □

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Conflicts of Interest

None.

Appendix A. Frobenius untwisting and a second proof of Proposition 2.4(2)

In this appendix, we have two goals: we first provide a proof of [Xia13, Theorem 6.1] in the context we need, which we use several times, and then we provide a second perspective on the termination of the Frobenius untwisting process in the context of Proposition 2.4.

Before we begin the proofs, we need one preliminary claim. Let $(\bar{\mathcal{C}}, \mathcal{Z})$ be a lift of $(\bar{C}, Z)$ over $W(k)$. Let $(\mathcal{V}, \nabla)$ be a vector bundle together with a logarithmic connection with nilpotent residues on $(\bar{\mathcal{C}}, \mathcal{Z})/W(k)$, such that ∇ is topologically quasi-nilpotent. (Therefore, $(\mathcal{V}, \nabla)$ is the *value* of a logarithmic crystal (C, Z) on the particular thickening $(\bar{\mathcal{C}}, \mathcal{Z})$).

CLAIM A.1. *The following two properties hold for* $(\mathcal{V}, \nabla)$.

(i) *The logarithmic isocrystal* $(\mathcal{V}, \nabla) \otimes \mathbb{Q}_p$ *over* $(\bar{\mathcal{C}}, \mathcal{Z})|_{W(k)[1/p]}$ *is semistable and of degree* 0.
(ii) *The degree of* $\mathcal{V}_p = \mathcal{V} \otimes \mathbb{F}_p$*, the restriction of the vector bundle* $\mathcal{V}$ *to* $\bar{C}$*, is* 0.

Proof of claim. To prove that $\mathcal{V} \otimes \mathbb{Q}_p$ has degree 0, it suffices to base change along a map $W(k)[1/p] \hookrightarrow \mathbb{C}$. Then the result follows from a computation of Esnault and Viehweg [EV86, Appendix B]. Now semistability follows easily. Indeed, any horizontal subsheaf of $\mathcal{V} \otimes \mathbb{Q}_p$ is necessarily a bundle, which is therefore equipped with a logarithmic flat connection and has nilpotent residues. By the first sentence, this implies that this horizontal subsheaf has degree 0, validating semistability.

To prove the second statement, it suffices to note that degree, being the first Chern class, is locally constant, see [KYZ20, § 6]; therefore, if the degree of $\mathcal{V} \otimes \mathbb{Q}_p$ is 0 on $\bar{\mathcal{C}}$, then so is the degree of $\mathcal{V}_p$ on $\bar{C}$. □

The third term in the following lemma is a special case of [Xia13, Theorem 6.1] that we need. In particular, we work in the context of strictly semistable p-divisible groups on $(\bar{C}, Z)$, as this allows us to discuss the destabilizing iteration.

LEMMA A.2. *Let* $G_C \to C$ *be a strictly semistable height* 2*, dimension* 1 p*-divisible group on* C*. Suppose the Kodaira–Spencer map of* $G_C \to C$ *is* 0.

(1) *Let* $\mathbb{D}(G_C)$ *be the Dieudonné module of* G_C*. Then the Dieudonné crystal* $\mathbb{D}(G_C)$ *canonically extends to a logarithmic Dieudonné crystal on* $(\bar{C}, Z)$.
(2) *Set* $(\mathcal{M}, \nabla, F, V)$ *denote the evaluation of the logarithmic extension of* $\mathbb{D}(G_C)$ *on the log pair* $(\bar{\mathcal{C}}, \mathcal{Z})$*. Then the Hodge line bundle* L *in* $\mathcal{M}_p = \mathcal{M} \otimes \mathbb{F}_p$ *has positive degree, which is the maximal destabilizing subbundle of* $(\mathcal{M}_p, \nabla_p)$.
(3) *Then there exists an isogenous* p*-divisible group* $G'_C \to C$*, such that the Frobenius pullback* ${G'_C}^{(p)}$ *is isomorphic to* G_C.

Proof. Since G_C is semistable, the first term follows from [Tri08, Corollary 3.14]. The second term follows from the existence of supersingular points as in Proposition 2.4(1) via the Hasse–Witt map. Consider the Kodaira–Spencer map

$$\theta : L \to \mathcal{M}_p / L \otimes \Omega^1_{\bar{C}}(\log Z).$$

By assumption $\theta = 0$, thus $L \subset (\mathcal{M}_p, \nabla_p)$ is a horizontal subbundle. Since $\deg \mathcal{M}_p = 0$ and $\deg L > 0$, the line bundle L is just the maximal destabilizing subbundle of $(\mathcal{M}_p, \nabla_p)$.

For the third term, We mainly follows Xia's original proof. Set $(\mathcal{M}', \nabla')$ to be the kernel of the following composition map

$$(\mathcal{M}, \nabla) \xrightarrow{\pi} (\mathcal{M}_p, \nabla_p) \to (\mathcal{M}_p, \nabla_p)/(L, \nabla_p) =: (N, \nabla_p),$$

where $\pi \colon (\mathcal{M}, \nabla) \to (\mathcal{M}_p, \nabla_p)$ is the reduction modulo p map. In particular, one has

$$p\mathcal{M} \subset \mathcal{M}', \quad \pi(\mathcal{M}') = \mathcal{M}'/p\mathcal{M} = L, \quad \text{and} \quad N = \mathcal{M}/\mathcal{M}'. \tag{A.1}$$

The crucial point is to show the Frobenius structure and the Verschiebung extend; if we show this, then we will obtain a new logarithmic Dieudonné module $(\mathcal{M}', \nabla', F', V')$.

Locally, over an affine open subset $\mathcal{U} = \mathrm{Spec}(R)$, we choose a lifting $\Phi \colon \widehat{R} \to \widehat{R}$ of the absolute Frobenius map $\sigma \colon R/pR \to R/pR$. Then the Frobenius structure and Verschiebung structure are given by

$$F \colon \mathcal{M}(U)^{\Phi} \to \mathcal{M}(U) \quad \text{and} \quad V \colon \mathcal{M}(U) \to \mathcal{M}(U)^{\Phi}.$$

Recall [dJ95, Proposition 2.5.2], L is the unique subbundle of $\mathcal{M}_p$ such that

$$(L)^\sigma = \mathrm{im}(V_p) = \ker(F_p). \tag{A.2}$$

In particular,

$$\pi\big(F(\mathcal{M}'(U)^\Phi)\big) = F_\mathfrak{p}\big(\pi(\mathcal{M}'(U)^\Phi)\big) = F_\mathfrak{p}\big(\pi(\mathcal{M}'(U))^\sigma\big) \overset{(A.4)}{=} F_p((L(U))^\sigma) \overset{(A.2)}{=} 0.$$

This implies that

$$F(\mathcal{M}'(U)^\Phi) \subseteq \mathfrak{p}\mathcal{M}(U) \subseteq \mathcal{M}'(U)$$

and that the Frobenius structure F can be restricted onto $(\mathcal{M}', \nabla')$, denoted by F'. Similarly,

$$\pi\big(V(\mathcal{M}'(U))\big) \subseteq \pi\big(V(\mathcal{M}(U))\big) = \mathrm{im}(V_p)(U) \overset{(A.2)}{=} (L)^\sigma. \tag{A.3}$$

This implies that

$$V(\mathcal{M}'(U)) \subseteq \pi^{-1}((L)^\sigma) \overset{(A.4)}{=} \mathcal{M}'(U)^\Phi$$

and that the Verschiebung structure V can be restricted onto $(\mathcal{M}', \nabla')$, denoted by V'. The module $(\mathcal{M}, \nabla, F, V)'$ is the realization of the $\mathbb{D}(G'_C)$ of a p-divisible group G'_C which satisfies ${G'_C}^{(p)} = G$ by de Jong's fundamental theorem. □

Second proof of Proposition 2.4(2). Recall that a p-divisible group is called *generically versally deformed* if the corresponding Kodaira–Spencer map is non-zero. From the third term in Lemma A.2, one may construct inductively:

(1) an infinite sequence of p-divisible groups over C

$$G_C^0 = G_C, G_C^1, G_C^2, \ldots$$

such that $(G_C^{i+1})^{(p)} = G_C^i$ for all $i \geq 0$ and whose Kodaira–Spencer maps are all zero; or

(2) a finite sequence of p-divisible groups

$$G_C^0 = G_C, G_C^1, \ldots, G_C^r$$

such that $(G_C^{i+1})^{(p)} = G_C^i$, the Kodaira–Spencer maps of G_C^i are zero for all $i \in \{0, \ldots, r-1\}$, and the Kodaira–Spencer map of G_C^r is non-zero.

To prove the second term of Proposition 2.4, one only need to show that the first case does not appear. Suppose we are in the first case. By a method of Langer in [Lan14, Theorem 5.1], we will construct a contradiction.

Let $(\mathcal{M}^i, \nabla^i, F^i, V^i)$ be the logarithmic Dieudonné module associated to G_C^i and let L^i be the Hodge line bundle in $\mathcal{M}_p^i$. According the construction of G_C^i as in the proof of Lemma A.2, one has

$$(\mathcal{M}^0, \nabla^0, F^0, V^0) \supsetneq (\mathcal{M}^1, \nabla^1, F^1, V^1) \supsetneq (\mathcal{M}^2, \nabla^2, F^2, V^2) \supsetneq \cdots,$$

where $(\mathcal{M}^{i+1}, \nabla^{i+1})$ to be the kernel of the following composition map

$$(\mathcal{M}^i, \nabla^i) \xrightarrow{\pi} (\mathcal{M}_p^i, \nabla_p^i) \to (\mathcal{M}_p^i, \nabla_p^i)/(L^i, \nabla_p^i) =: (N^i, \nabla_p^i),$$

where $\pi\colon (\mathcal{M}^i, \nabla^i) \to (\mathcal{M}_p^i, \nabla_p^i)$ is the reduction modulo p map. In particular, one has

$$p\mathcal{M}^i \subset \mathcal{M}^{i+1}, \quad \pi(\mathcal{M}^{i+1}) = \mathcal{M}^{i+1}/p\mathcal{M}^i = L^i, \quad \text{and} \quad N^i = \mathcal{M}^i/\mathcal{M}^{i+1}. \tag{A.4}$$

Denote

$$(\overline{\mathcal{M}}^m, \overline{\nabla}^m, \overline{F}^m, \overline{V}^m) := \varprojlim_{n \geq m} (\mathcal{M}, \nabla, F, V)^m/(M, \nabla, F, V)^n.$$

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In the following, we show that the generic fiber $\overline{\mathcal{M}}^m \otimes K$ of $\overline{\mathcal{M}}^m$ is a destabilizing quotient of $\mathcal{M}^m \otimes K = \mathcal{M}_K$ for sufficiently large $m \gg 0$; this will contradict semistability of $\mathcal{M}_K$.

From (A.4) and the construction of the sequence, one has exact sequences of $\mathcal{O}_{\mathfrak{X}_\mathfrak{p}}$-modules

$$0 \to L^n \to \mathcal{M}_p^n \to N^n \to 0$$

and

$$0 \to N^n \xrightarrow{p} \mathcal{M}_p^{n+1} \to L^n \to 0.$$

Denote by C^n the kernel of the composition $L^{n+1} \to M_p^{n+1} \to L^n$. By the same reason as in the proof of [Lan14, Theorem 5.1], $C^n = 0$ for sufficient large n. By eliminating the first finitely many terms, we may assume $C^n = 0$ for all $n \geq 0$. Thus, one gets injections

$$L^0 \supseteq L^1 \supseteq L^2 \supseteq \cdots \quad \text{and} \quad N^0 \subseteq N^1 \subseteq N^2 \subseteq \cdots.$$

Since the slope of L^n is non-increasing and also non-negative, the sequence L^n stabilizes to some L. Since $\deg L^n + \deg N^n = \deg \mathcal{M}_\mathfrak{p}^n = 0$, the sequence N^n stabilizes to some N. Once again, by eliminating the first finitely many terms, we may assume $L^n = L$ and $N^n = N$ for all $n \geq 0$. In particular, the composition map

$$\mathcal{M}^n/\mathcal{M}^{n+1} = N^n \xrightarrow{p} \mathcal{M}_\mathfrak{p}^{n+1} \to N^{n+1} = \mathcal{M}^{n+1}/\mathcal{M}^{n+2}$$

is an isomorphism for all $n \geq 0$. Thus,

$$\mathcal{M}^{n+1} = p\mathcal{M}^n + \mathcal{M}^{n+2} \tag{A.5}$$

and

$$p\mathcal{M}^{n+1} = p\mathcal{M}^n \cap \mathcal{M}^{n+2} \tag{A.6}$$

for each $n \geq 0$. We show that $\mathcal{M}^i = p^i\mathcal{M}^0 + \mathcal{M}^n$ for any $0 \leq i \leq n$ and $p\mathcal{M}^{n-1} = p\mathcal{M}^0 \cap \mathcal{M}^n$ as follows:

$$\begin{aligned} \mathcal{M}^i &\overset{(A.5)}{=} p\mathcal{M}^{i-1} + \mathcal{M}^{i+1} \overset{(A.5)}{=} p\mathcal{M}^{i-1} + p\mathcal{M}^{i+1} + \mathcal{M}^{i+2} \\ &\overset{(A.5)}{=} p\mathcal{M}^{i-1} + p\mathcal{M}^{i+1} + \cdots + p\mathcal{M}^{n-1} + \mathcal{M}^n = p\mathcal{M}^{i-1} + \mathcal{M}^n, \end{aligned} \tag{A.7}$$

$$\begin{aligned} \mathcal{M}^i &\overset{(A.7)}{=} p\mathcal{M}^{i-1} + \mathcal{M}^n = \underbrace{p(p(\cdots p(p}_{i}\mathcal{M}^0 + \mathcal{M}^n) + \mathcal{M}^n)\cdots + \mathcal{M}^n) + \mathcal{M}^n \\ &= p^i\mathcal{M}^0 + \mathcal{M}^n, \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} p\mathcal{M}^{n-1} &\overset{(A.6)}{=} p\mathcal{M}^{n-2} \cap \mathcal{M}^n \overset{(A.6)}{=} (p\mathcal{M}^{n-3} \cap \mathcal{M}_{n-1}) \cap \mathcal{M}^n \\ &\overset{(A.6)}{=} ((\cdots(p\mathcal{M}^0 \cap \mathcal{M}_1) \cap \cdots) \cap \mathcal{M}_{n-1}) \cap \mathcal{M}^n = p\mathcal{M}^0 \cap \mathcal{M}^n. \end{aligned} \tag{A.9}$$

In particular, there is an $W_n = W(k)/p^n$ module structure on $\mathcal{M}^0/\mathcal{M}^n$. Consider

$$\begin{aligned} (\mathcal{M}^0/\mathcal{M}^n) \otimes_{W_n} (pW_n/p^nW_n) &\cong (\mathcal{M}^0/\mathcal{M}^n) \otimes_{W_n} W_n/p^{n-1} \\ &= \mathcal{M}^0/(p^{n-1}\mathcal{M}^0 + \mathcal{M}^n) = \mathcal{M}^0/\mathcal{M}^{n-1} \\ &\cong p\mathcal{M}^0/p\mathcal{M}^{n-1} = p\mathcal{M}^0/(p\mathcal{M}^0 \cap \mathcal{M}^n) \\ &\cong (p\mathcal{M}^0 + \mathcal{M}^n)/\mathcal{M}^n = \mathcal{M}^1/\mathcal{M}^n, \end{aligned}$$

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where the map can be easily checked to be given by $\overline{m}\otimes\overline{a}\mapsto\overline{am}$ for any $m\in\mathcal{M}^0$ and any $a\in p$. One also has

$$(\mathcal{M}^0/\mathcal{M}^n)\otimes_{W_n}k=\mathcal{M}^0/(p\mathcal{M}^0+\mathcal{M}^n)=\mathcal{M}^0/\mathcal{M}^1,$$

which implies the following sequence is exact

$$0\to(\mathcal{M}^0/\mathcal{M}^n)\otimes_{W(k)}pW_n/p^n\to(\mathcal{M}^0/\mathcal{M}^n)\otimes_W W(k)/p^n\to(\mathcal{M}^0/\mathcal{M}^n)\otimes_{\mathcal{O}_K}k\to 0.$$

Thus, $\mathrm{Tor}^{W/p^n}(k,\mathcal{M}^0/\mathcal{M}^n)=0$ and $\mathcal{M}^0/\mathcal{M}^n$ is flat over W_n. It follows that $\widetilde{\mathcal{M}}^0=\varprojlim_{n\geq 0}\mathcal{M}^0/\mathcal{M}^n$ is a $W(k)$-flat coherent $\mathcal{O}_{\bar{\mathcal{C}}}$-module having a filtration with quotients isomorphic to N. Thus, $\widetilde{\mathcal{M}}^0\otimes W(k)[1/p]$ is a destabilizing quotient of $\mathcal{M}^0\otimes W(k)[1/p]$, which contradicts the semistability of $\mathcal{M}^0\otimes W(k)[1/p]$. □

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